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A SPECIAL CASE OF A CONJECTURE BY WIDOM WITH IMPLICATIONS TO FERMIONIC ENTANGLEMENT ENTROPY

ROBERT HELLING, HAJO LESCHKE, AND WOLFGANG SPITZER

ABSTRACT. We prove a special case of a conjecture in asymptotic analysis by Harold Widom. More precisely, we establish the leading and next-to-leading term of a semi-classical expansion of the trace of the square of certain integral operators on the Hilbert space $L^2(\mathbb{R}^d)$. As already observed by Gioev and Klich, this implies that the bi-partite entanglement entropy of the free Fermi gas in its ground state grows at least as fast as the surface area of the spatially bounded part times a logarithmic enhancement.

1. INTRODUCTION

In contrast to systems of classical physics, a quantum system composed of two distinguishable parts may be in a pure state which is not a product of pure states of its subsystems. Consequently, if the total system is in such an *entangled* pure state, the partial state of each subsystem is not pure, in other words, mixed. Following Einstein, Schrödinger, Bell, and others, entanglement may be used to rule out interpretations of quantum mechanics which are both local and realistic, similar to those of classical (statistical) mechanics. More recently, entanglement has been established as a key concept of quantum communication and information theory. For example, quantum teleportation and quantum computing heavily rely on it [2, 23, 31, 40].

Partially triggered by the latter theories quantifications of entanglement (e.g., in terms of entropy) and consequences thereof are at present intensively discussed for states of many-particle systems. We refer to the reviews by Amico, Fazio, Osterloh, and Vedral [1], and by Peschel and Eisler [33]. Here, several interesting results and conjectures were put forward. However, in most cases a mathematical proof is not yet available and one relies on heuristic arguments, approximate calculations and/or numerical observations. This is even true for the entanglement entropy of the ground states of quantum spin-chains (see Vidal, Latorre, Ricco, and Kitaev [44]) and of a system being as simple as the free Fermi gas. Since in the latter system there is no interaction at all, a non-trivial entanglement entropy is solely due to the effective coupling of the particles by the Fermi-Dirac statistics, the algebraic statement of Pauli's exclusion principle.

The interest in entanglement entropy was also sparked from quantum field theory, and in particular by toy models for the Bekenstein-Hawking entropy of black holes [5]. Srednicki [38] and Bombelli, Koul, Lee, and Sorkin [9] found numerically that in a semi-classical limit which corresponds to scaling the bounded region Ω in d -dimensional Euclidean position space \mathbb{R}^d by $R > 0$ and taking $R \rightarrow \infty$, the bi-partite entanglement entropy is not a bulk property but scales with the area $R^{d-1}|\partial\Omega|$ of the boundary surface rather than the volume $R^d|\Omega|$. This so-called *area law* is thought to be generic for field theories with a spectral gap above the ground-state energy. See also the more recent works by Cramer, Eisert, and Plenio [17] and by Cramer and Eisert [16] who proved the area-law scaling for harmonic lattice systems. It has been suggested that entanglement might be the mechanism behind the black-hole entropy. At first, Bekenstein and Hawking found that black holes behave thermally if one interprets the surface gravity as temperature and the area of the horizon as entropy. Especially,

there is a “second law” which states that in physical processes the total horizon area can never decrease. It is a major challenge for a quantum theory of gravity to show that this is not merely an analogy but that the area of the horizon is indeed proportional to physical entropy and to give a microscopic explanation thereof. In the framework of string theory this was achieved for extremal black holes by Strominger and Vafa [39] and Maldacena and Strominger [29]. More generally, it has been argued that the entanglement entropy scales as R^{d-1} for $d \geq 2$ space dimensions, while for $d = 1$ one expects a *logarithmic* scaling, $\ln R$. In a theory with correlation length $\xi < \infty$, heuristic arguments suggest that the entanglement entropy stems from correlations across $\partial\Omega$ in a layer of width $\xi > 0$ and the absence of long-range correlations is responsible for the area law. However, the area law is observed in conformal field theories for $1+1$ space-time dimensions as well, where $\xi = \infty$, see Calabrese and Cardy [13, 14].

Coming back to simple fermionic systems, Jin and Korepin [25] showed for the first time that for free fermions on the one-dimensional lattice \mathbb{Z} the entanglement entropy for $R\Omega = [-R, R] \cap \mathbb{Z}$ indeed scales as $\ln R$, see also Fannes, Haegeman, and Mosonyi [18]. Wolf [50] and later Farkas and Zimboras [19] then proved for $d \geq 2$ and cubic $\Omega \subset \mathbb{Z}^d$ a lower bound on the partial particle-number variance that scales as $R^{d-1} \ln R$. This, in turn, implies that the entanglement entropy grows at least as fast as $R^{d-1} \ln R$, thereby ruling out an area law. Barthel, Chung and Schollwöck [4] and independently Li, Ding, Yu, Roscilde, and Haas [28] provided numerical support that the entropy itself scales in the same way up to a numerical factor (for $d = 2$ and $d = 3$).

To our knowledge, Gioev and Klich [21] were the first to observe an intimate connection between the scaling of the entanglement entropy of the free-Fermi-gas ground state and an important conjecture in asymptotic analysis by Harold Widom [47–49]. This “Widom conjecture” concerns a two-term asymptotic expansion of the trace, $\text{tr } F(A)$, for a wide class of analytic functions F of certain integral operators A on the Hilbert space $L^2(\mathbb{R}^d)$, see Equation (4) below. The conjecture may be understood as a multi-dimensional generalization of Szegő’s asymptotics for Toeplitz determinants and of Slepian’s spectral asymptotics in classical information theory on the capacity of a communication channel which is band limited in both frequency and time. In a similar vein, Gioev [20] established, among other things, for the ground state of the free Fermi gas in \mathbb{R}^d and rather general $\Omega \subset \mathbb{R}^d$ with smooth $\partial\Omega$ a lower bound on the partial particle-number variance that scales as $R^{d-1} \ln R$. The main result of the present paper establishes an $R^{d-1} \ln R$ behavior of that variance itself and provides the precise pre-factor in terms of a simple surface integral times a numerical constant. Our result is, in fact, a proof of a special case of the Widom conjecture for quadratic F .

Although we only have a lower bound on the entanglement entropy of the free-Fermi-gas ground state, we believe, in accordance with a conjecture by Gioev and Klich [21], that this bound reflects the correct scaling of the entropy itself up to a numerical factor, which is independent of the Fermi sea $\Gamma \subset \mathbb{R}^d$ characterizing the ground state, and the region $\Omega \subset \mathbb{R}^d$. As they pointed out, their conjecture actually goes beyond the Widom conjecture because the entropy corresponds to a non-analytic function F (see our Remark 6(iv)). Regardless of the validity of the Gioev–Klich conjecture, their works [20, 21] were key stimuli to us and apparently also to the authors of [4, 28].

The structure of the present paper is as follows: In the next section we formulate the Widom conjecture. In Section 3 we prove the Widom conjecture for quadratic polynomials F . Then we proceed in Section 4 to compile some background material on fermionic entanglement entropy and apply our result. In Section 5 we give an outlook of how to possibly prove the Widom conjecture for arbitrary polynomials. The paper ends with appendices on the method of stationary phase, on the decay properties of certain Fourier integrals, and on a simple extension of Roccaforte’s estimate on the volume of certain self-intersections [35].

After having finished the first version of this paper we have learned from Alexander Sobolev [37] that he has a proof of the Widom conjecture for all polynomials, F , based on pseudo-differential-operator calculus. We are grateful for the explanation of his remarkable achievement prior to publication.

2. THE “QUADRATIC” WIDOM CONJECTURE

We start with some notation used throughout the paper. If $d \geq 2$, we denote a vector $\mathbf{v} \in \mathbb{R}^d$ by a boldface letter, and write $v := |\mathbf{v}| := (\mathbf{v} \cdot \mathbf{v})^{1/2}$ for its norm. Here, we use a dot to denote the Euclidean scalar product $\mathbf{v} \cdot \mathbf{w}$ of two vectors \mathbf{v}, \mathbf{w} in \mathbb{R}^d . By $A+B := \{\mathbf{a}+\mathbf{b} : \mathbf{a} \in A, \mathbf{b} \in B\}$ we denote the arithmetic (or Minkowski) sum of a pair of subsets $A, B \subseteq \mathbb{R}^d$. We also write $A+\mathbf{b} := A+\{\mathbf{b}\}$ for $A \subseteq \mathbb{R}^d$ translated by $\mathbf{b} \in \mathbb{R}^d$ and $RA := \{R\mathbf{a} : \mathbf{a} \in A\}$ for $A \subseteq \mathbb{R}^d$ multiplied by $R \in \mathbb{R}$. For a Borel set $\Lambda \subseteq \mathbb{R}^d$ we denote its volume with respect to the d -dimensional Lebesgue measure as $|\Lambda| := \int_{\Lambda} d\mathbf{x} = \int_{\mathbb{R}^d} d\mathbf{x} \chi_{\Lambda}(\mathbf{x})$, where χ_{Λ} stands for the indicator function of Λ . In particular, if Λ is the positive half-line, $\Theta := \chi_{[0, \infty[}$ denotes the right-continuous Heaviside unit-step function. The Hilbert space of complex-valued, Lebesgue square-integrable functions $f : \Lambda \rightarrow \mathbb{C}$ is denoted as usual by $L^2(\Lambda)$. We use the Bachmann–Landau notation of “little oh” and “big Oh” in asymptotic (in)equalities in the sense that for real-valued functions f, g on \mathbb{R} , we write

- $f(R) \geq g(R) + o(h(R))$ if $\lim_{R \rightarrow \infty} \frac{f(R)-g(R)}{h(R)} \geq 0$;
- $f(R) = g(R) + o(h(R))$ if $\lim_{R \rightarrow \infty} \frac{f(R)-g(R)}{h(R)} = 0$;
- $f(R) = g(R) + O(h(R))$ if $\limsup_{R \rightarrow \infty} \left| \frac{f(R)-g(R)}{h(R)} \right| < \infty$.

Next, we formulate our basic assumption.

ASSUMPTION 1. *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ and $\Gamma \subset \mathbb{R}^d$ be $(C^\infty-)$ smooth, compact, d -dimensional manifolds-with-boundary. The orientation of Ω and of its boundary surface $\partial\Omega$ is the one induced from \mathbb{R}^d , respectively, from the manifold, and similarly for Γ and $\partial\Gamma$. Let α be a smooth, complex-valued function on an open set in $\mathbb{R}^d \times \mathbb{R}^d$ containing $\Omega \times \Gamma$.*

Note that for such an Ω also the difference $\Omega - \Omega := \Omega + (-1)\Omega$ is a smooth, compact manifold-with-boundary. For background material in (Riemannian) differential geometry we refer to the textbooks [7, 10, 43] without further notice.

For two sets Ω, Γ , and a function α as described in Assumption 1 we define for each $R > 0$ the integral operator $A_R : L^2(\Omega) \rightarrow L^2(\Omega)$ by its kernel

$$a_R(\mathbf{x}, \mathbf{y}) := \left(\frac{R}{2\pi} \right)^d \int_{\Gamma} d\mathbf{p} e^{iR(\mathbf{x}-\mathbf{y}) \cdot \mathbf{p}} \alpha(\mathbf{x}, \mathbf{p}) \quad (1)$$

in the sense that

$$(A_R f)(\mathbf{x}) := \int_{\Omega} d\mathbf{y} a_R(\mathbf{x}, \mathbf{y}) f(\mathbf{y}), \quad \mathbf{x} \in \Omega, f \in L^2(\Omega). \quad (2)$$

Because of

$$\int_{\Omega \times \Omega} d\mathbf{x}_1 d\mathbf{x}_2 |a_R(\mathbf{x}_1, \mathbf{x}_2)|^2 \leq \left(\frac{R}{2\pi} \right)^{2d} |\Omega|^2 |\Gamma|^2 \|\alpha\|_{\infty, \Omega, \Gamma}^2, \quad (3)$$

where $\|\alpha\|_{\infty, \Omega, \Gamma} := \sup\{|\alpha(\mathbf{x}, \mathbf{p})| : (\mathbf{x}, \mathbf{p}) \in \Omega \times \Gamma\} < \infty$, the operator A_R is in the Hilbert–Schmidt class, see [34, Theorem VI.23]. By [34, Theorem VI.22(h)], the square A_R^2 (and consequently each natural power $A_R^k, k \geq 3$) is then a trace-class operator.

We recall that A_R can be trivially extended to an operator on $L^2(\mathbb{R}^d)$ by viewing $L^2(\Omega)$ as a subspace of $L^2(\mathbb{R}^d)$ and considering $\widehat{\chi_{\Omega}} A_R \widehat{\chi_{\Omega}}$, where the multiplication operator $\widehat{\chi_{\Omega}}$ is the orthogonal projection from $L^2(\mathbb{R}^d)$ to $L^2(\Omega)$. The operators $\widehat{\chi_{\Omega}} A_R \widehat{\chi_{\Omega}}$ and A_R have the same non-zero eigenvalues with the same multiplicities. Therefore, if F is a complex-valued function with $F(0) = 0$ and being analytic on a disc centered at the origin and with radius strictly larger than $\|\alpha\|_{\infty, \Omega, \Gamma}$, then $\text{tr } F(A_R) = \text{tr } F(\widehat{\chi_{\Omega}} A_R \widehat{\chi_{\Omega}})$.

In this and similar situations, Widom [48, 49] conjectured the beautiful two-term asymptotic expansion

$$\begin{aligned} \operatorname{tr} F(A_R) &= \left(\frac{R}{2\pi}\right)^d \int_{\Omega \times \Gamma} d\mathbf{x} d\mathbf{p} F(\alpha(\mathbf{x}, \mathbf{p})) \\ &+ \left(\frac{R}{2\pi}\right)^{d-1} \ln R \int_{\partial\Omega \times \partial\Gamma} d\sigma(\mathbf{x}) d\sigma(\mathbf{p}) |\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}| \tilde{F}(\alpha(\mathbf{x}, \mathbf{p})) + o(R^{d-1} \ln R). \end{aligned} \quad (4)$$

Here, the linear transformation $F \mapsto \tilde{F}$ is defined by

$$\tilde{F}(\xi) := \frac{1}{4\pi^2} \int_0^1 dt \frac{F(t\xi) - tF(\xi)}{t(1-t)}, \quad \xi \in \mathbb{R}, \quad (5)$$

$\mathbf{n}_{\mathbf{x}} \in \mathbb{R}^d$ and $\mathbf{n}_{\mathbf{p}} \in \mathbb{R}^d$ denote the outward unit normal vector at $\mathbf{x} \in \partial\Omega$, respectively at $\mathbf{p} \in \partial\Gamma$, and σ is the canonical $(d-1)$ -dimensional area measure on the boundary surfaces $\partial\Omega$ and $\partial\Gamma$.

Actually, Widom [49] proved (4) in the case that Γ is a half-space and Ω is compact with smooth $\partial\Omega$. In [35], Roccaforte considered the case $\Gamma = \mathbb{R}^d$ and convolution operators arising from a function α not depending on \mathbf{x} , and whose Fourier transform is decaying sufficiently fast. Remarkably, he proved a three-term asymptotic expansion $aR^d + bR^{d-1} + cR^{d-2} + o(R^{d-2})$ of $\operatorname{tr} F(A_R)$ for certain analytic functions F and identified the coefficients a, b , and c from geometric properties of Ω .

If $F(t) = t$ and if A_R is a trace-class operator then one simply has,

$$\operatorname{tr} F(A_R) = \operatorname{tr} A_R = \left(\frac{R}{2\pi}\right)^d \int_{\Omega \times \Gamma} d\mathbf{x} d\mathbf{p} \alpha(\mathbf{x}, \mathbf{p}) = \left(\frac{R}{2\pi}\right)^d \int_{\Omega \times \Gamma} d\mathbf{x} d\mathbf{p} F(\alpha(\mathbf{x}, \mathbf{p})) \quad (6)$$

for all $R > 0$. Our main result is the special case of (4) with $F(t) = t^2$ as $R \rightarrow \infty$.

THEOREM 2 (“Quadratic” Widom Conjecture). *Under Assumption 1, the two-term asymptotic expansion*

$$\begin{aligned} \operatorname{tr} (A_R)^2 &= \left(\frac{R}{2\pi}\right)^d \int_{\Omega \times \Gamma} d\mathbf{x} d\mathbf{p} \alpha(\mathbf{x}, \mathbf{p})^2 - \frac{1}{4\pi^2} \left(\frac{R}{2\pi}\right)^{d-1} \ln R \int_{\partial\Omega \times \partial\Gamma} d\sigma(\mathbf{x}) d\sigma(\mathbf{p}) |\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}| \alpha(\mathbf{x}, \mathbf{p})^2 \\ &+ o(R^{d-1} \ln R) \end{aligned} \quad (7)$$

holds as $R \rightarrow \infty$.

- REMARKS 3.**
- (i) For dimension $d = 1$ and with Ω and Γ compact intervals, formula (7) remains true as can be seen by an explicit computation. In this case, the surface integral is simply the sum of the four values taken by the function α at the four corners of the rectangle $\Omega \times \Gamma$.
 - (ii) Our proof relies on the method of stationary phase (see [41]) and an expression for the volume of the intersection of a set with its translate as an integral over the boundary (see Theorem 12 in Appendix B). The proof is elementary in the sense that it does not rely on tools from pseudo-differential-operator calculus used by Widom [49] and recently by Sobolev [37] (see the end of the Introduction).
 - (iii) In definition (1) one could evaluate the “phase-space function” (or “symbol”) α more generally at $(\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}), \mathbf{p})$ with $\lambda \in [0, 1]$ instead of choosing $\lambda = 0$. The resulting “ λ -quantization of α ” would then lead to an operator $A_{R,\lambda}$ [30]. Here, $1/R$ plays the role of Planck’s constant. It can be seen from our proof of Theorem 2 that the asymptotic behavior of $\operatorname{tr} A_{R,\lambda}^2$ as $R \rightarrow \infty$ has the same leading term and next-to-leading term as $\operatorname{tr} A_R^2$ for all $\lambda \in [0, 1]$.

3. PROOF OF THEOREM 2

The proof consists of two parts. The first part deals with the leading term proportional to R^d and an error term of the order R^{d-1} . In the second part we show how the term proportional to $R^{d-1} \ln R$ emerges.

We start out with a simple change of co-ordinates, $\mathbf{u} := \mathbf{x}_1$, $\mathbf{v} := \mathbf{x}_1 - \mathbf{x}_2$, scale \mathbf{v} by $1/R$, and hence write [27, p. 524],

$$\mathrm{tr}(A_R)^2 = \int_{\Omega \times \Omega} d\mathbf{x}_1 d\mathbf{x}_2 a_R(\mathbf{x}_1, \mathbf{x}_2) a_R(\mathbf{x}_2, \mathbf{x}_1) \quad (8)$$

$$= \int_{\Omega - \Omega} d\mathbf{v} \int_{\Omega} d\mathbf{u} a_R(\mathbf{u}, \mathbf{u} - \mathbf{v}) a_R(\mathbf{u} - \mathbf{v}, \mathbf{u}) \quad (9)$$

$$= \left(\frac{R}{4\pi^2}\right)^d \int_{R(\Omega - \Omega)} d\mathbf{v} \int_{\Gamma \times \Gamma} d\mathbf{p} d\mathbf{q} e^{i\mathbf{v} \cdot (\mathbf{p} - \mathbf{q})} \times \int_{\mathbb{R}^d} d\mathbf{u} \alpha(\mathbf{u}, \mathbf{p}) \alpha(\mathbf{u} - \mathbf{v}/R, \mathbf{q}) \chi_{\Omega}(\mathbf{u}) \chi_{\Omega}(\mathbf{u} - \mathbf{v}/R). \quad (10)$$

First we expand $\alpha(\mathbf{u} - \mathbf{v}/R, \mathbf{q})$ at (\mathbf{u}, \mathbf{q}) . The error term is $O(1/R)$. The integral over \mathbf{u} is then of the form ($\varepsilon = R^{-1}$)

$$\int_{\Omega \cap (\Omega + \varepsilon \mathbf{v})} d\mathbf{u} f(\mathbf{u}) = \int_{\Omega} d\mathbf{u} f(\mathbf{u}) - \int_{\Omega \setminus (\Omega \cap (\Omega + \varepsilon \mathbf{v}))} d\mathbf{u} f(\mathbf{u}) \quad (11)$$

with $f(\mathbf{u}) := \alpha(\mathbf{u}, \mathbf{p}) \alpha(\mathbf{u}, \mathbf{q})$.

Let us define for each $\mathbf{x} \in \Omega$ the function $\gamma_{\mathbf{x}} : \mathbb{R}^d \rightarrow \mathbb{C}$, $\mathbf{v} \mapsto \gamma_{\mathbf{x}}(\mathbf{v})$ by

$$\gamma_{\mathbf{x}}(\mathbf{v}) := (2\pi)^{-d} \int_{\Gamma} d\mathbf{p} \alpha(\mathbf{x}, \mathbf{p}) e^{i\mathbf{v} \cdot \mathbf{p}}. \quad (12)$$

Then using the uniform decay, $\sup_{\mathbf{x} \in \Omega} |\gamma_{\mathbf{x}}(\mathbf{v})| \leq C v^{-\frac{d+1}{2}}$ (see Lemma 11), and Parseval’s identity we obtain for the “leading term”,

$$\left| \int_{R(\Omega - \Omega)} d\mathbf{v} \int_{\Gamma \times \Gamma} d\mathbf{p} d\mathbf{q} \alpha(\mathbf{u}, \mathbf{p}) \alpha(\mathbf{u}, \mathbf{q}) e^{i\mathbf{v} \cdot (\mathbf{p} - \mathbf{q})} - (2\pi)^d \int_{\Gamma} d\mathbf{p} \alpha(\mathbf{u}, \mathbf{p})^2 \right| \leq C/R. \quad (13)$$

For the second term in equation (11) we use Theorem 12 with $\varepsilon = R^{-1}$. Then, after a change of variables we have to analyze the integral

$$I := \left(\frac{R}{2\pi}\right)^{2d} \int_{\partial\Omega} d\sigma(\mathbf{x}) \int_{\Omega - \Omega} d\mathbf{v} \max(0, \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}}) \gamma_{\mathbf{x}}(R\mathbf{v}) \gamma_{\mathbf{x}}(-R\mathbf{v}), \quad (14)$$

and a remainder term (proportional to v^2) which is easy to deal with using the decay of $\gamma_{\mathbf{x}}$. Namely,

$$R^{2d} \left| \int_{\Omega - \Omega} d\mathbf{v} v^2 \gamma_{\mathbf{x}}(R\mathbf{v}) \gamma_{\mathbf{x}}(-R\mathbf{v}) \right| \leq C R^{d-1} \quad (15)$$

for some constant C . In order to continue with I from (14), it is convenient to write¹ $\max(0, \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}}) = \Theta(\mathbf{v} \cdot \mathbf{n}_{\mathbf{x}}) \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}}$. Integrating by parts we get

$$(2\pi)^d \mathbf{v} \gamma_{\mathbf{x}}(R\mathbf{v}) = \frac{1}{iR} \int_{\Gamma} d\mathbf{p} \alpha(\mathbf{x}, \mathbf{p}) \frac{\partial}{\partial \mathbf{p}} e^{iR\mathbf{v} \cdot \mathbf{p}} \quad (16)$$

$$= \frac{1}{iR} \left(\int_{\partial\Gamma} d\sigma(\mathbf{p}) \mathbf{n}_{\mathbf{p}} \alpha(\mathbf{x}, \mathbf{p}) e^{iR\mathbf{v} \cdot \mathbf{p}} - \int_{\Gamma} d\mathbf{p} \left(\frac{\partial}{\partial \mathbf{p}} \alpha(\mathbf{x}, \mathbf{p}) \right) e^{iR\mathbf{v} \cdot \mathbf{p}} \right). \quad (17)$$

For the second integral (over Γ) we may once more integrate by parts and deduce that it is a term of lower order by another factor of R^{-1} . Therefore, for some constant C ,

$$\begin{aligned} & \left| I + i \left(\frac{R}{2\pi}\right)^d R^{-1} \int_{\partial\Omega \times \partial\Gamma} d\sigma(\mathbf{x}) d\sigma(\mathbf{p}) \mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}} \alpha(\mathbf{x}, \mathbf{p}) \right. \\ & \quad \left. \times R^d \int_{\Omega - \Omega} d\mathbf{v} \Theta(\mathbf{v} \cdot \mathbf{n}_{\mathbf{x}}) \gamma_{\mathbf{x}}(-R\mathbf{v}) e^{iR\mathbf{v} \cdot \mathbf{p}} \right| \leq C R^{d-1}. \end{aligned} \quad (18)$$

¹Recall that Θ is the Heaviside function.

The hard part is to analyze the last integral and show that for σ -almost each $\mathbf{p} \in \partial\Gamma$ one has

$$\left| R^d \int_{\Omega-\Omega} d\mathbf{v} \Theta(\mathbf{v} \cdot \mathbf{n}_{\mathbf{x}}) \gamma_{\mathbf{x}}(-R\mathbf{v}) e^{iR\mathbf{v} \cdot \mathbf{p}} + (2\pi i)^{-1} \operatorname{sgn}(\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}) \alpha(\mathbf{x}, \mathbf{p}) \ln R \right| = o(R). \quad (19)$$

Here, we need the precise asymptotic expansion (69) for $\gamma_{\mathbf{x}}(\mathbf{v})$ from Lemma 11 and the notation employed in its proof. For the resulting phase $\mathbf{v} \mapsto R(\mathbf{v} \cdot \mathbf{p} - \mathbf{v} \cdot \mathbf{k})$ we are going to apply once more the method of stationary phase. To this end, we recall from Lemma 11 that each \mathbf{k} implicitly depends on \mathbf{v} and use generalized polar co-ordinates, (ρ, \mathbf{w}) , to perform the \mathbf{v} -integration over $\Omega - \Omega$. In general, $\mathbf{w} \in \partial(\Omega - \Omega)$ and $\rho \in [0, 1]$ are not independent of each other and ρ does not necessarily cover the whole interval $[0, 1]$. Nevertheless, we may consider the full interval $[0, 1]$ by counting contributions with a negative sign if at $\mathbf{w} \in \partial(\Omega - \Omega)$ the boundary is “inwards” in the sense that $\mathbf{w} \cdot \mathbf{n}_{\mathbf{w}}$ is negative. This is sketched in Figure 1. It can be seen that even though $\rho\mathbf{w}$ is not necessarily in $\Omega - \Omega$, points outside are counted with total weight zero while points inside are counted with total weight one.

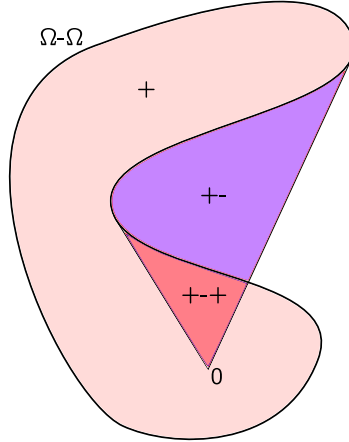


FIGURE 1. The integration over $\Omega - \Omega$.

Let us pick an ortho-normal basis in \mathbb{R}^d such that the normal vector $\mathbf{n}_{\mathbf{p}}$ to $\partial\Gamma$ at the given point $\mathbf{p} \in \partial\Gamma$ points in the d -th direction. Locally around \mathbf{p} , let $\partial\Gamma$ be given by the graph of a function $f : U_f \rightarrow \mathbb{R}, \mathbf{t} \mapsto f(\mathbf{t})$ with some open $U_f \subset \mathbb{R}^{d-1}$; in the notation used in the proof of Lemma 11, $f = f(\mathbf{n}_{\mathbf{p}}, m)$ and $U_f = U_{\mathbf{n}_{\mathbf{p}}, m}$ for some m . We write $\mathbf{p} = (\mathbf{s}, f(\mathbf{s}))$ for some $\mathbf{s} \in U_f$ and note that, without loss of generality (by appealing to Sard's Theorem), f is not only critical but has an extremum at \mathbf{s} .

In a similar fashion, we can locally write $\partial(\Omega - \Omega)$ as the graph of another function $g : U_g \rightarrow \mathbb{R}, \mathbf{u} \mapsto g(\mathbf{u})$ with some open $U_g \subset \mathbb{R}^{d-1}$. We assume for the moment that $\Omega - \Omega$ is convex, or put differently that all boundary points are outwards. Then, we partition the integration in (19) into cones $V := \{(\rho\mathbf{u}, \rho g(\mathbf{u})) : \rho \in [0, 1], \mathbf{u} \in U_g\} \subseteq \Omega - \Omega$ with some open set $U_g \subset \mathbb{R}^{d-1}$. Furthermore, instead of integrating ρ over $[0, 1]$ we may integrate over $[C/R, 1]$ for some constant C without changing the leading asymptotics of the integral as $R \rightarrow \infty$.

The crucial step is to express the condition (see Lemma 10(i)) $\mathbf{e} := \mathbf{v}/v = \operatorname{sgn}(\mathbf{v} \cdot \mathbf{k}) \mathbf{n}_{\mathbf{k}}$ in these new co-ordinates, where, without loss of generality, $\mathbf{k} = (\mathbf{t}_{\mathbf{k}}, h(\mathbf{t}_{\mathbf{k}}))$ with $h = f^{(\mathbf{e}, m')}$ for some m' . $\mathbf{t}_{\mathbf{k}}$ is now thought of as a function of \mathbf{u} . Note that $\mathbf{n}_{\mathbf{k}} = \operatorname{sgn}(\mathbf{n}_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{k}})(-\frac{\partial}{\partial \mathbf{t}} h(\mathbf{t}_{\mathbf{k}}), 1)/\sqrt{1 + |\frac{\partial}{\partial \mathbf{t}} h(\mathbf{t}_{\mathbf{k}})|^2}$

and thus

$$\begin{aligned}
 \frac{(\mathbf{u}, g(\mathbf{u}))}{\sqrt{\mathbf{u}^2 + g(\mathbf{u})^2}} &= \operatorname{sgn}(\mathbf{v} \cdot \mathbf{n}_k) \operatorname{sgn}(\mathbf{n}_p \cdot \mathbf{n}_k) \frac{(-\frac{\partial}{\partial \mathbf{t}} h(\mathbf{t}_k), 1)}{\sqrt{1 + |\frac{\partial}{\partial \mathbf{t}} h(\mathbf{t}_k)|^2}} \\
 &= \operatorname{sgn}(g(\mathbf{u})) \frac{(-\frac{\partial}{\partial \mathbf{t}} h(\mathbf{t}_k), 1)}{\sqrt{1 + |\frac{\partial}{\partial \mathbf{t}} h(\mathbf{t}_k)|^2}}.
 \end{aligned} \tag{20}$$

Let us proceed with the situation that $(\mathbf{0}, g(\mathbf{0})) \in V$ and call this cone V_0 . As there might be several disjoint graphs of $f(\mathbf{n}_p, m)$, it is notationally simpler to use their union and call the corresponding function again f . This amounts to setting $h = f$.

The volume element reads $d\mathbf{v} = \rho^{d-1} f(\mathbf{u}) d\rho d\mathbf{u}$. As \mathbf{v} is parallel to $\mathbf{n}_p = (0, \dots, 0, 1)$ at $\mathbf{u} = \mathbf{0}$ and V_0 can be chosen small enough, we have $\Theta(\mathbf{v} \cdot \mathbf{n}_x) = \Theta(\operatorname{sgn}(\mathbf{v} \cdot \mathbf{n}_p) \mathbf{n}_p \cdot \mathbf{n}_x)$, and the only contribution to the integral is from those \mathbf{v} where $g(\mathbf{u})$ has the same sign as the last component of \mathbf{n}_x . Using the asymptotics from Lemma 11 we find up to lower-order terms that (using the abbreviation $\mathcal{K}_u := \mathcal{K}_{(\mathbf{u}, g(\mathbf{u}))/\sqrt{\mathbf{u}^2 + g(\mathbf{u})^2}}$)

$$\begin{aligned}
 R^d \int_{V_0} d\mathbf{v} \Theta(\mathbf{v} \cdot \mathbf{n}_x) \gamma_x(-R\mathbf{v}) e^{iR\mathbf{v} \cdot \mathbf{p}} \\
 = i(2\pi)^{-\frac{d+1}{2}} \int_C d\rho \rho^{-\frac{d+3}{2}} \int_{U_g} d\mathbf{u} \frac{g(\mathbf{u})}{(\|\mathbf{u}\|^2 + g(\mathbf{u})^2)^{(d+1)/4}} \\
 \times \sum_{\mathbf{k} \in \mathcal{K}_u} \frac{\Theta((\mathbf{u}, g(\mathbf{u})) \cdot \mathbf{n}_x) \operatorname{sgn}((\mathbf{u}, g(\mathbf{u})) \cdot \mathbf{n}_k)}{\sqrt{|\det(f_{ij}(\mathbf{t}_k))|}} \alpha(\mathbf{x}, R(\mathbf{t}_k, f(\mathbf{t}_k))) \\
 \times \exp[iR\rho(\mathbf{u}, g(\mathbf{u})) \cdot (\mathbf{p} - \mathbf{k})].
 \end{aligned} \tag{21}$$

Let us write the last \mathbf{u} -integral in the form $\int_{U_g} d\mathbf{u} \psi(\mathbf{u}) \exp[iR\rho\phi(\mathbf{u})]$. Then we smoothly extend ψ to a compactly supported complex-valued function $\tilde{\psi}$ on \mathbb{R}^{d-1} and ϕ to compactly supported real-valued function $\tilde{\phi}$ on \mathbb{R}^{d-1} in such a way that $\tilde{\phi}$ does not acquire new critical points on the support of $\tilde{\psi}$ outside the support of ψ . By Proposition 8, this does not change the leading asymptotics of the integral.

Let us investigate now the critical points of the phase function $\phi : U_g \subset \mathbb{R}^{d-1} \rightarrow \mathbb{R}, \mathbf{u} \mapsto (\mathbf{u}, g(\mathbf{u})) \cdot (\mathbf{p} - \mathbf{k}(\mathbf{u}))$. Taking derivatives of both sides of equation (20) and evaluating at $\mathbf{u} = \mathbf{0}$ yields (we use the sum convention and sum over indices that appear twice)

$$\frac{du_i}{|g(\mathbf{0})|} = -\operatorname{sgn}(g(\mathbf{0})) \frac{\partial^2 f}{\partial t_i \partial t_j}(\mathbf{t}_k(\mathbf{0})) dt_j. \tag{22}$$

This implies

$$\frac{\partial t_j}{\partial u_i}(\mathbf{0}) = -\frac{f_{ij}^{-1}(\mathbf{t}_k(\mathbf{0}))}{g(\mathbf{0})}, \tag{23}$$

where f_{ij}^{-1} denotes the matrix inverse of the Hessian of f . We are now ready to expand the phase to second order in \mathbf{u} at $\mathbf{0}$:

$$\begin{aligned}
& \mathbf{v} \cdot (\mathbf{p} - \mathbf{k}) / \rho \\
&= \mathbf{u} \cdot (\mathbf{s} - \mathbf{t}_{\mathbf{k}}(\mathbf{u})) + g(\mathbf{u})(f(\mathbf{s}) - f(\mathbf{t}_{\mathbf{k}}(\mathbf{u}))) \\
&= g(\mathbf{0})(f(\mathbf{s}) - f(\mathbf{t}_{\mathbf{k}}(\mathbf{0}))) \\
&\quad + u_i \left(s_i - (\mathbf{t}_{\mathbf{k}}(\mathbf{0}))_i + \frac{\partial g}{\partial u_i}(\mathbf{0})(f(\mathbf{s}) - f(\mathbf{t}_{\mathbf{k}}(\mathbf{0}))) - g(\mathbf{0}) \frac{\partial f}{\partial t_j}(\mathbf{t}_{\mathbf{k}}(\mathbf{0})) \frac{\partial t_j}{\partial u_i}(\mathbf{0}) \right) \\
&\quad + u_i u_j \left(-\frac{\partial t_i}{\partial u_j}(\mathbf{0}) + \frac{1}{2} \frac{\partial^2 g}{\partial u_i \partial u_j}(\mathbf{0})(f(\mathbf{s}) - f(\mathbf{t}_{\mathbf{k}}(\mathbf{0}))) - \frac{1}{2} g(\mathbf{0}) \frac{\partial^2 f}{\partial t_r \partial t_l}(\mathbf{t}_{\mathbf{k}}(\mathbf{0})) \frac{\partial t_r}{\partial u_i} \frac{\partial t_l}{\partial u_j}(\mathbf{0}) \right. \\
&\quad \left. - \frac{\partial g}{\partial u_i}(\mathbf{0}) \frac{\partial f}{\partial t_r}(\mathbf{t}_{\mathbf{k}}(\mathbf{0})) \frac{\partial t_r}{\partial u_j}(\mathbf{0}) - \frac{1}{2} g(\mathbf{0}) \frac{\partial f}{\partial t_r}(\mathbf{t}_{\mathbf{k}}(\mathbf{0})) \frac{\partial^2 t_r}{\partial u_i \partial u_j}(\mathbf{0}) \right) \\
&\quad + O(u^3).
\end{aligned} \tag{24}$$

Using (23) we obtain

$$\begin{aligned}
& \mathbf{v} \cdot (\mathbf{p} - \mathbf{k}) / \rho \\
&= g(\mathbf{0})(f(\mathbf{s}) - f(\mathbf{t}_{\mathbf{k}}(\mathbf{0}))) + \mathbf{u} \cdot (\mathbf{s} - \mathbf{t}_{\mathbf{k}}(\mathbf{0})) + \frac{1}{2} u_i u_j \left(\frac{f_{ij}^{-1}(\mathbf{t}_{\mathbf{k}}(\mathbf{0}))}{g(\mathbf{0})} + g_{ij}(\mathbf{0})(f(\mathbf{s}) - f(\mathbf{t}_{\mathbf{k}}(\mathbf{0}))) \right) \\
&\quad + O(u^3).
\end{aligned} \tag{25}$$

Then for V_0 small enough, the only critical point of the phase function ϕ is when $\mathbf{t}_{\mathbf{k}}(\mathbf{0}) = \mathbf{s}$ and hence $\mathbf{k} = \mathbf{p}$. In this case,

$$\phi(\mathbf{u}) = \rho \frac{f_{ij}^{-1}(\mathbf{t}_{\mathbf{k}}(\mathbf{0}))}{2g(\mathbf{0})} u_i u_j + O(u^3). \tag{26}$$

Next, we apply Proposition 9 and conclude that asymptotically (up to next-to-leading terms in $1/R$)

$$\begin{aligned}
& R^d \int_{V_0} d\mathbf{v} \Theta(\mathbf{v} \cdot \mathbf{n}_{\mathbf{x}}) \gamma_{\mathbf{x}}(-R\mathbf{v}) e^{iR\mathbf{v} \cdot \mathbf{p}} \\
&= i(2\pi)^{-\frac{d+1}{2}} \int_C^R d\rho \rho^{-1} g(\mathbf{0})^{-\frac{d-1}{2}} \frac{\text{sgn}(\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}})}{\sqrt{|\det(f_{ij}(\mathbf{s}))|}} \alpha(\mathbf{x}, \mathbf{p}) \\
&\quad \times \exp \left[-\frac{i\pi}{4} \text{sgn}(f_{ij}(\mathbf{s})) \right] \int_{\mathbb{R}^{d-1}} d\mathbf{u} \exp \left[-i \frac{f_{ij}^{-1}(\mathbf{s})}{2g(\mathbf{0})} u_i u_j \right] \\
&= (2\pi i)^{-1} \text{sgn}(\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}) \alpha(\mathbf{x}, \mathbf{p}) \int_C^R d\rho \rho^{-1} \\
&= (2\pi i)^{-1} \text{sgn}(\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}) \alpha(\mathbf{x}, \mathbf{p}) \ln R.
\end{aligned} \tag{27}$$

Now we address the other cones in the polar decomposition of $\Omega - \Omega$ and show that they contribute to lower order. We choose a parametrization g of the boundary, and cast $\mathbf{w} \in \partial(\Omega - \Omega)$ in the form $\mathbf{w} = (\mathbf{u}, g(\mathbf{u}))$. Let $\phi : \partial(\Omega - \Omega) \rightarrow \mathbb{R}, \mathbf{w} \mapsto \mathbf{w} \cdot (\mathbf{p} - \mathbf{k}(\mathbf{w}))$ be the phase function in the integral (19) in the co-ordinate \mathbf{w} . We may assume that $\mathbf{k}(\mathbf{w}) \neq \mathbf{p}$. Since $\mathbf{w} \cdot T_{\mathbf{w}} \mathbf{k} = 0$, a point \mathbf{w}_0 is a critical point of ϕ if and only if $\mathbf{p} - \mathbf{k}(\mathbf{w}_0)$ is parallel to $\mathbf{n}_{\mathbf{w}_0}$. In addition, we require that $\phi(\mathbf{w}_0) = 0$, which means that $\mathbf{p} - \mathbf{k}(\mathbf{w}_0)$ is perpendicular to \mathbf{w}_0 . For if $\phi(\mathbf{w}_0) \neq 0$, then the ρ -integral $\int_C^R d\rho / \rho \exp[i\rho\phi(\mathbf{w}_0)]$ would be only of the order 1. For a convex hypersurface $\partial(\Omega - \Omega)$ these two conditions on \mathbf{w}_0 cannot be satisfied at once and we conclude that only the cone V_0 gives the leading contribution. In the case of a general hypersurface $\partial(\Omega - \Omega)$, such cases have to be dismissed by the fact they form a set of zero surface measure.

Finally, we discuss the case that $\Omega - \Omega$ is not convex. By the above argument we only need to consider a cone in the direction of $\mathbf{n}_{\mathbf{p}}$. Instead of the local parametrization of the boundary surface $\partial(\Omega - \Omega)$ as the graph of a single function g we have, in general, a finite number, M , of functions g_n defined on open sets U_{g_n} for $1 \leq n \leq M$. We may order them so that $U_{g_n} \subset U_{g_{n+1}}$

and that $\text{graph}(g_M)$ is the furthest part of the boundary surface, and thus the normal vector points outwards. Let us define the cones $V_0^{(n)} := \{(\rho \mathbf{u}, \rho g_n(\mathbf{u})) : \rho \in [0, 1], \mathbf{u} \in U_{g_n}\}$ and let us repeat the above calculation for each of these cone. We count their (asymptotic) contributions, namely $(2\pi i)^{-1} \text{sgn}(\mathbf{n}_x \cdot \mathbf{n}_p) \alpha(\mathbf{x}, \mathbf{p}) \ln R$, positive/negative if the normal vector at $\partial(\Omega - \Omega) \cap \text{graph}(g_n)$ is outwards/inwards. Since 0 is always in the interior of $\Omega - \Omega$ and hence M is odd there is only one such term that survives this summation and we have finished the proof. \square

REMARK 4. We have assumed that the boundaries $\partial\Omega$ and $\partial\Gamma$ are smooth. We believe that our proof extends to the case of C^3 -boundaries. Some regularity, however, is needed as can be seen from the example of cubes $\Omega = \Gamma = [-1, 1]^d$ with $\alpha = 1$. In this case the Fourier transform γ , defined in (57), is simply given by the product

$$\gamma(\mathbf{v}) = \prod_{i=1}^d \frac{\sin(v_i)}{\pi v_i} \quad (28)$$

with $\mathbf{v} = (v_1, \dots, v_d)$. Hence, the leading decay of $\gamma(\mathbf{v})$ for large $|\mathbf{v}|$ is of the form $|\mathbf{v}|^{-n}$ with $n \in \{1, \dots, d\}$ depending on the direction $\mathbf{v}/|\mathbf{v}|$. This is in contrast to the leading decay $|\mathbf{v}|^{-(d+1)/2}$ in case of a smooth $\partial\Gamma$. However, the average decay of the Fourier transform is still of the order $|\mathbf{v}|^{-(d+1)/2}$ as was proved by Brandolini, Hofmann, and Iosevich [11] for convex sets. In our proof of Theorem (2) we critically use the decay behavior for domains Γ fulfilling our Assumption 1. It is obvious that cubes are not covered. However, for the above example of cubes, (7) can be proved by a direct computation (cf. Remark 3(i)).

4. FERMIONIC ENTANGLEMENT ENTROPY

We are going to apply Theorem 2 to the ground state of the free Fermi gas in the infinitely extended position space \mathbb{R}^d . To fix our notation and to supply some background material we first consider a slightly more general situation.

4.1. Entanglement entropy of quasi-free fermionic states. A general system of many fermionic particles with separable one-particle Hilbert space \mathcal{H} with its scalar product denoted by $\langle \cdot, \cdot \rangle$, is described by the (smallest) C^* -algebra ([12, Theorem 5.2.5], [6, Section 4.2]) $\mathcal{A}_{\mathcal{H}}$ generated by the unit operator 1 and the annihilation and creation operators $a(f)$ and $a^*(g)$ for all $f, g \in \mathcal{H}$. These operators are bounded and satisfy the usual canonical anti-commutation relations,

$$a(f)a^*(g) + a^*(g)a(f) = \langle f, g \rangle 1, \quad f, g \in \mathcal{H}, \quad (29)$$

$$a(f)a(g) + a(g)a(f) = 0. \quad (30)$$

A state ρ is a linear functional $\rho : \mathcal{A}_{\mathcal{H}} \rightarrow \mathbb{C}$ with $\rho(1) = 1$ and $\rho(X^*X) \geq 0$ for all $X \in \mathcal{A}_{\mathcal{H}}$. A state ρ is called *quasi-free* (and *gauge-invariant*) [12, p. 43] if there exists a self-adjoint operator D on \mathcal{H} with $0 \leq D \leq 1$, such that

$$\rho(a^*(f)a(g)) = \langle g, Df \rangle, \quad (31)$$

and, more generally,

$$\rho(a^*(f_1) \cdots a^*(f_m)a(g_1) \cdots a(g_n)) = \begin{cases} 0 & \text{if } m \neq n \\ \det \langle g_i, Df_j \rangle & \text{if } m = n \end{cases} \quad (32)$$

for all finite sets $\{f_1, \dots, f_m, g_1, \dots, g_n\} \subset \mathcal{H}$. In this sense a quasi-free ρ is a *generalized Gaussian* state, where D plays the role of the covariance. We call D the *one-particle density operator* characterizing ρ . We note that ρ is *pure* if and only if D is a projection, that is, $D^2 = D$.

In order to define the (von Neumann) entropy of a quasi-free state we first introduce the function

$$\eta(t) := \begin{cases} 0 & \text{if } t \in \{0, 1\} \\ -t \ln t - (1-t) \ln(1-t) & \text{if } t \in]0, 1[\end{cases}. \quad (33)$$

Now, if $\eta(D)$ is a trace-class operator then the (von Neumann) *entropy*, $S(\rho)$, of the quasi-free ρ characterized by D may be defined as (see [32, Equation (6.9)]),

$$S(\rho) := \text{tr } \eta(D). \quad (34)$$

It follows that the entropy of a quasi-free state ρ is zero if and only if ρ is pure; this equivalence remains true for non quasi-free states but we refrain here from defining the entropy for general states.

For a general state ρ and an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into two closed subspaces \mathcal{H}_1 and \mathcal{H}_2 we use the isomorphism $\mathcal{A}_{\mathcal{H}} \cong \mathcal{A}_{\mathcal{H}_1} \otimes \mathcal{A}_{\mathcal{H}_2}$ to define two *partial* (marginal or reduced) states ρ_1 and ρ_2 on $\mathcal{A}_{\mathcal{H}_1}$ and $\mathcal{A}_{\mathcal{H}_2}$, respectively, by

$$\rho_1(X) := \rho(X \otimes 1), \quad X \in \mathcal{A}_{\mathcal{H}_1}, \quad (35)$$

$$\rho_2(X) := \rho(1 \otimes X), \quad X \in \mathcal{A}_{\mathcal{H}_2}. \quad (36)$$

Then one has the “triangle” inequality comprised of the Araki–Lieb inequality [3] and the subadditivity of entropy [32, Theorem 6.15],

$$|S(\rho_1) - S(\rho_2)| \leq S(\rho) \leq S(\rho_1) + S(\rho_2). \quad (37)$$

Here, the left-hand side is zero by definition if $S(\rho_1) = S(\rho_2) = \infty$. As a consequence of (37), the *partial entropies* $S(\rho_1)$ and $S(\rho_2)$ are equal if the (total) state ρ is pure. A simple quantification of the correlations between the subsystems corresponding to \mathcal{H}_1 and \mathcal{H}_2 in the state ρ of the total system, not present in the product state $\rho_1 \otimes \rho_2$, is the (bi-partite) *entanglement entropy*,

$$\Delta S(\rho) := S(\rho_1 \otimes \rho_2) - S(\rho) = S(\rho_1) + S(\rho_2) - S(\rho) \geq 0. \quad (38)$$

For a pure state ρ this simplifies to

$$\Delta S(\rho) = 2S(\rho_1) = 2S(\rho_2). \quad (39)$$

In words, for a pure state the entanglement entropy is just twice its partial entropies.

If ρ is quasi-free, then ρ_1 and ρ_2 are quasi-free, too. More precisely, if ρ is characterized by D on \mathcal{H} as above, then ρ_ℓ ($\ell \in \{1, 2\}$) is characterized by the partial one-particle density operator

$$D_\ell := E_\ell D E_\ell, \quad (40)$$

where $E_\ell : \mathcal{H} \rightarrow \mathcal{H}_\ell$ is the orthogonal projection from \mathcal{H} onto \mathcal{H}_ℓ . Since D_1 (resp. D_2) is the zero-operator on \mathcal{H}_2 (resp. \mathcal{H}_1) it is naturally identified with an operator on \mathcal{H}_1 (resp. \mathcal{H}_2). By construction, the following identities hold,

$$\rho_\ell(a^*(f)a(g)) = \langle g, D_\ell f \rangle, \quad \text{etc. (in analogy to (32))} \quad (41)$$

$$S(\rho_\ell) = \text{tr } \eta(D_\ell). \quad (42)$$

In the special case that the mean of the total number of particles is finite, that is, $\text{tr } D < \infty$, then the state ρ is given [12, Theorem 5.2.14 & pp. 36–37] by a density operator W on the fermionic Fock space $\mathcal{F}(\mathcal{H})$ over \mathcal{H} . This positive operator of unit trace may be written as

$$W = \det(1 - D) \exp \left[- \sum_{n,m} \langle f_n, \ln(D^{-1} - 1)f_m \rangle a^*(f_n)a(f_m) \right], \quad (43)$$

where $\{f_n\}$ is an arbitrary orthonormal basis of \mathcal{H} . Then one has [42, 2.5, 11, p. 401], [46]

$$S(\rho) = \text{tr } \eta(D) = -\text{tr } W \ln W, \quad (44)$$

which motivates our definition (34). We stress that $\text{tr } D < \infty$ is not sufficient for $\text{tr } \eta(D) < \infty$ if \mathcal{H} has infinite dimension. Conversely, the example $D = 1$ shows that $\text{tr } \eta(D) = 0 < \infty$ is possible although $\text{tr } D = \infty$.

In the case that (only) $\text{tr } D_1 < \infty$, then (at least) ρ_1 uniquely corresponds to a density operator W_1 on $\mathcal{F}(\mathcal{H}_1)$ given by a formula analogous to (43). Accordingly, one then has

$$S(\rho_1) = \text{tr } \eta(D_1) = -\text{tr } W_1 \ln W_1. \quad (45)$$

Sometimes it is convenient to consider besides the von Neumann entropy also a more general (but not subadditive) entropy dating back to Rényi. More precisely, if $\text{tr } D_1 < \infty$, we define the partial Rényi entropy of order β as (cf. [45, Section II.G])

$$S_\beta(\rho_1) := \frac{1}{1-\beta} \ln \text{tr } W_1^\beta, \quad \beta \in]0, \infty[\setminus \{1\}. \quad (46)$$

Note that $S_\beta(\rho_1) \geq 0$ and $\lim_{\beta \rightarrow 1} S_\beta(\rho_1) = S(\rho_1)$. Moreover, the Jensen inequality implies the monotonicity,

$$(\beta - \beta') (S_{\beta'}(\rho_1) - S_\beta(\rho_1)) \geq 0. \quad (47)$$

It may be viewed as a special case of an inequality between (fractional) absolute moments of a random variable, dating back at least to a work of Schlömilch in 1858, see [22, p. 26]. In analogy to (45) the quasi-free nature of ρ_1 implies

$$S_\beta(\rho_1) = \text{tr } \eta_\beta(D_1), \quad (48)$$

where

$$\eta_\beta(t) := \frac{1}{1-\beta} \ln(t^\beta + (1-t)^\beta), \quad t \in [0, 1]. \quad (49)$$

For later use we also mention the chain of estimates

$$\begin{aligned} 2 \text{tr } D_1(1 - D_1) &\leq S_2(\rho_1) \leq (4 \ln 2) \text{tr } D_1(1 - D_1) \leq S(\rho_1) \\ &\leq S_{1/2}(\rho_1) \leq 2 \text{tr } D_1^{1/2}(1 - D_1)^{1/2} \leq 2 \text{tr } D_1^{1/2}. \end{aligned} \quad (50)$$

The first three estimates follow from $2t(1-t) \leq \eta_2(t) \leq (4 \ln 2)t(1-t) \leq \eta(t)$ if $t \in [0, 1]$. The fourth one is (47) with $\beta = 1/2$ and $\beta' \rightarrow 1$, and the last two follow from $\eta_{1/2}(t) \leq 2t^{1/2}(1-t)^{1/2} \leq 2t^{1/2}$.

We now see that $\text{tr } D_1^{1/2} < \infty$ is not only sufficient for $\text{tr } D_1 < \infty$ but also for $\text{tr } \eta(D_1) < \infty$.

While $\text{tr } D_1$ is physically interpreted as the mean of the number of particles, the quantity $\text{tr } D_1(1 - D_1)$ occurring in (50) is the variance of that number in the quasi-free state ρ_1 of the subsystem corresponding to \mathcal{H}_1 .

4.2. Entanglement entropy of the ground state of the free Fermi gas. Now we consider the special case of a free, spinless Fermi gas in d -dimensional Euclidean space \mathbb{R}^d , $d \in \mathbb{N}$, at zero absolute temperature, that is, in its ground state. In the terminology of Section 4.1 this state ρ is quasi-free and characterized by the Fermi projection $D = \Theta(\mu - H)$ on $\mathcal{H} = L^2(\mathbb{R}^d)$. Here, $H = h(-i\frac{\partial}{\partial \mathbf{x}})$ is a translation-invariant one-particle Hamiltonian given in terms of a smooth “dispersion” function, $h : \mathbb{R}^d \rightarrow \mathbb{R}$, on momentum space, which tends to infinity near infinity and ensures that H is a self-adjoint operator on \mathcal{H} . The prime example is $h(\mathbf{p}) = p^2$, corresponding to the non-relativistic kinetic energy (in the absence of a magnetic field). The real parameter $\mu > \inf\{h(\mathbf{p}) : \mathbf{p} \in \mathbb{R}^d\}$ is the Fermi energy. Obviously, one has $\text{tr } D = \infty$ but $S(\rho) = \text{tr } \eta(D) = 0 < \infty$ due to $D^2 = D$. The Fermi sea corresponding to the Fermi projection is given as the lower level set

$$\Gamma = \{\mathbf{p} \in \mathbb{R}^d : h(\mathbf{p}) \leq \mu\} \quad (51)$$

in momentum space.

In order to study the finite-volume properties of the Fermi gas we consider a Borel set $\Omega \subset \mathbb{R}^d$ with finite volume $|\Omega|$ and thus choose $\mathcal{H}_1 = L^2(\Omega)$ and $\mathcal{H}_2 = L^2(\mathbb{R}^d \setminus \Omega)$. Then, according to Section 4.1, the partial state $\rho_1 =: \rho_\Omega$ of that part of the Fermi gas with bounded position space Ω is quasi-free and characterized by

$$D_1 = \widehat{\chi_\Omega} \Theta(\mu - H) \widehat{\chi_\Omega} =: D_\Omega. \quad (52)$$

We may therefore identify D_Ω with the operator A_1 defined in (2) with the function $\alpha = 1$ and Γ given by (51). Moreover, one has (cf. [27, p. 524] for the calculation of $\text{tr } D_\Omega$)

$$\text{tr } D_\Omega^2 \leq \text{tr } D_\Omega = (2\pi)^{-d} |\Omega| |\Gamma| < \infty, \quad (53)$$

and, by (50), even

$$S(\rho_\Omega) = \text{tr } \eta(D_\Omega) \leq 2 \text{tr } D_\Omega^{1/2} < \infty. \quad (54)$$

Here, the finiteness of $\text{tr } D_\Omega^{1/2}$ and hence that of the partial entropy $S(\rho_\Omega)$ of the free Fermi gas in its (pure) ground state ρ , was proved by Gioev and Klich [21] by using certain decay properties of singular values due to Birman and Solomyak [8] (see also Chang and Ha [15]). We mention in passing that the mean particle *density*, $\text{tr } D_\Omega / |\Omega| = |\Gamma| / (2\pi)^d$ is a non-decreasing function of μ .

Theorem 2 has the following

COROLLARY 5 (Lower bound on fermionic entropy). *Suppose Γ of (51) and Ω satisfy Assumption 1. Then, the partial entropy $S(\rho_{R\Omega})$ of the free-Fermi-gas ground state satisfies the asymptotic inequality*

$$S(\rho_{R\Omega}) \geq \frac{\ln 2}{\pi^2} \left(\frac{R}{2\pi} \right)^{d-1} \ln R \int_{\partial\Omega \times \partial\Gamma} d\sigma(\mathbf{x}) d\sigma(\mathbf{p}) |\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}| + o(R^{d-1} \ln R). \quad (55)$$

REMARKS 6. (i) It was already observed by Gioev and Klich [21] that a proof of (7) with the function $\alpha = 1$ would imply (55). As mentioned in the Introduction, Gioev [20, Inequalities (1.8) & (1.9)] has previously established a smaller lower bound on $S(\rho_{R\Omega})$ with the same $R^{d-1} \ln R$ -scaling.

(ii) An important consequence of the $R^{d-1} \ln R$ -scaling of the leading term in (55) is that it rules out an area law for the entanglement entropy in the sense that $\liminf_{R \rightarrow \infty} \frac{2S(\rho_{R\Omega})}{R^{d-1}} = \infty$. The μ -dependence of that term is encoded in the Fermi surface $\partial\Gamma$.

(iii) Gioev and Klich [20, 21] also provided an upper bound on $S(\rho_{R\Omega})$ which is, however, larger by an extra factor $\ln R$. No smaller upper bound is known to us.

(iv) One may also consider the partial Rényi entropies $S_\beta(\rho_{R\Omega})$. For instance, if $\beta = 2$, then (50) gives lower and upper bounds on $S_2(\rho_{R\Omega})$ in terms of the partial particle-number variance $\text{tr } D_{R\Omega}(1 - D_{R\Omega})$, which both scale as $R^{d-1} \ln R$. More generally, by an informal application of the Widom conjecture (4) with $\alpha = 1$ to the (non-analytic) function $F = \eta_\beta$ from (49) and using $\tilde{\eta}_\beta(1) = (1 + \beta)/(24\beta)$ it is tempting to conjecture the exact leading asymptotic behavior of the partial Rényi entropy of order β to be

$$S_\beta(\rho_{R\Omega}) = \frac{1 + \beta}{24\beta} \left(\frac{R}{2\pi} \right)^{d-1} \ln R \int_{\partial\Omega \times \partial\Gamma} d\sigma(\mathbf{x}) d\sigma(\mathbf{p}) |\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}| + o(R^{d-1} \ln R). \quad (56)$$

The von Neumann limit $\beta \rightarrow 1$ of (56) has already been conjectured by Gioev and Klich [21] and has stimulated the authors of [4, 28]. To our knowledge, the validity of (56) is open even for $d = 1$ and compact intervals Ω and Γ (cf. Remark 3(i)). See, however, Jin and Korepin [25, Equation (4)] for non-interacting fermions on the one-dimensional lattice \mathbb{Z} .

Proof of Corollary 5. In the (conventional) definition (2) of the operator A_R one keeps Ω fixed and (effectively) scales Γ by R . Here, we need to interchange the roles of the two sets since physically the ground state of the Fermi gas in \mathbb{R}^d , and hence its Fermi sea Γ is fixed. And one wants to understand the asymptotic growth of the entanglement entropy with increasing volume $|\Omega|$ of the position space Ω . The required interchangeability is justified by the fact that the two products QPQ and PQP in terms of two arbitrary orthogonal projection operators Q and P (on $L^2(\mathbb{R}^d)$) have the same non-zero eigenvalues with the same multiplicities. This follows from the singular-value decompositions of QP and PQ , see e.g. [36, Section 1.2]. Using the third inequality in (50) for a lower bound, recalling from (53) that $\text{tr } A_R = \text{tr } D_{R\Omega} = (\frac{R}{2\pi})^d |\Omega| |\Gamma|$, and applying Theorem 2 with $\alpha = 1$ finally gives (55). \square

5. OUTLOOK

Now we show a possible route towards a proof of the Widom conjecture for polynomials of arbitrary degree. The reader will have noticed that the essential difficulty is already present for the special case $\alpha = 1$, and that the extension to general α is rather straightforward. In what follows we will therefore put $\alpha = 1$. Then $\gamma_{\mathbf{x}}(\mathbf{v})$ of (12) reduces to the simple Fourier integral,

$$\gamma(\mathbf{v}) := (2\pi)^{-d} \int_{\Gamma} d\mathbf{p} e^{i\mathbf{v} \cdot \mathbf{p}}, \quad \mathbf{v} \in \mathbb{R}^d. \quad (57)$$

It reproduces itself under convolution, that is, $\gamma * \gamma = \gamma$, reflecting the identity $\chi_\Gamma^2 = \chi_\Gamma$.

Proceeding as in equation (8) we write for $k \in \mathbb{N}$

$$\text{tr } (A_R)^k = \int_{\mathbb{R}^{kd}} \prod_{j=1}^k d\mathbf{x}_j \gamma(\mathbf{x}_j - \mathbf{x}_{j+1}) \chi_{R\Omega}(\mathbf{x}_j), \quad \mathbf{x}_{k+1} := \mathbf{x}_1, \quad (58)$$

and introduce new co-ordinates $\mathbf{y}_0 := \mathbf{x}_1, \mathbf{y}_1 := \mathbf{x}_2 - \mathbf{x}_1, \dots, \mathbf{y}_{k-1} := \mathbf{x}_k - \mathbf{x}_{k-1}$. Note that $\mathbf{y}_0 \in R\Omega, \mathbf{y}_1 \in R\Omega - \mathbf{y}_0, \dots, \mathbf{y}_{k-1} \in R\Omega - \mathbf{y}_0 - \dots - \mathbf{y}_{k-2}$. Then

$$\begin{aligned} \text{tr}(A_R)^k &= \int_{\mathbb{R}^{(k-1)d}} d\mathbf{y}_1 \cdots d\mathbf{y}_{k-1} \gamma(-\mathbf{y}_1) \cdots \gamma(-\mathbf{y}_{k-1}) \gamma(\mathbf{y}_1 + \dots + \mathbf{y}_{k-1}) \\ &\quad \times \int_{\mathbb{R}^d} d\mathbf{y}_0 \chi_{R\Omega}(\mathbf{y}_0) \chi_{R\Omega}(\mathbf{y}_0 + \mathbf{y}_1) \cdots \chi_{R\Omega}(\mathbf{y}_0 + \dots + \mathbf{y}_{k-1}). \end{aligned} \quad (59)$$

For the last integral we write (using Lemma 12)

$$\int_{\mathbb{R}^d} d\mathbf{y}_0 \chi_{R\Omega}(\mathbf{y}_0) \chi_{R\Omega}(\mathbf{y}_0 + \mathbf{y}_1) \cdots \chi_{R\Omega}(\mathbf{y}_0 + \dots + \mathbf{y}_{k-1}) = \quad (60)$$

$$\begin{aligned} &= R^d |\Omega| - R^d |\Omega \setminus (\Omega \cap (\Omega - \mathbf{y}_1/R) \cap \dots \cap (\Omega - \mathbf{y}_1/R - \dots - \mathbf{y}_{k-1}/R))| \\ &= \left[R^d |\Omega| - R^{d-1} \int_{\partial\Omega} d\sigma(\mathbf{x}) \max(0, \mathbf{y}_1 \cdot \mathbf{n}_\mathbf{x}, \dots, (\mathbf{y}_1 + \dots + \mathbf{y}_{k-1}) \cdot \mathbf{n}_\mathbf{x}) \right. \\ &\quad \left. + O(R^{d-2}) \right] \times \chi_{R(\Omega-\Omega)}(\mathbf{y}_1) \cdots \chi_{R(\Omega-\Omega)}(\mathbf{y}_1 + \dots + \mathbf{y}_{k-1}). \end{aligned} \quad (61)$$

For the leading term in (58) we obtain $(\frac{R}{2\pi})^d |\Omega| |\Gamma|$ by the same argument as for $k = 2$ with an error $O(R^{d-1})$; recall that $\gamma * \dots * \gamma = \gamma$.

Since in (59) the function $(\mathbf{y}_1, \dots, \mathbf{y}_{k-1}) \mapsto \gamma(-\mathbf{y}_1) \cdots \gamma(-\mathbf{y}_{k-1}) \gamma(\mathbf{y}_1 + \dots + \mathbf{y}_{k-1})$ is symmetric we only need to consider the symmetric part of the remaining function, namely of

$$\begin{aligned} (\mathbf{y}_1, \dots, \mathbf{y}_{k-1}) &\mapsto \max(0, \mathbf{y}_1 \cdot \mathbf{n}_\mathbf{x}, \dots, (\mathbf{y}_1 + \dots + \mathbf{y}_{k-1}) \cdot \mathbf{n}_\mathbf{x}) \\ &\quad \times \chi_{R(\Omega-\Omega)}(\mathbf{y}_1) \cdots \chi_{R(\Omega-\Omega)}(\mathbf{y}_1 + \dots + \mathbf{y}_{k-1}). \end{aligned} \quad (62)$$

The maximum function by itself can be easily symmetrized by the following quite surprising combinatorial lemma.

LEMMA 7. *Let a_1, \dots, a_n be real numbers. Then*

$$\sum_{\sigma} \max(0, a_{\sigma 1}, a_{\sigma 1} + a_{\sigma 2}, \dots, a_{\sigma 1} + \dots + a_{\sigma n}) = \sum_{\sigma} \sum_{\ell=1}^n \frac{1}{\ell} \max(0, a_{\sigma 1} + \dots + a_{\sigma \ell}), \quad (63)$$

where on both sides the summation \sum_{σ} runs over the $n!$ permutations of $\{1, \dots, n\} \subset \mathbb{N}$.

The lemma was formulated and used in this version by Widom [47, pp. 171, 174]. Under the same assumptions, Kac [26, Theorem 4.2] presents a proof (due to F. Dyson) that

$$\sum_{\sigma} \max(0, a_{\sigma 1}, a_{\sigma 1} + a_{\sigma 2}, \dots, a_{\sigma 1} + \dots + a_{\sigma n}) = \sum_{\sigma} a_{\sigma 1} \sum_{k=1}^n \Theta(a_{\sigma 1} + \dots + a_{\sigma k}), \quad (64)$$

where Θ is (as above) the Heaviside function. It can be easily shown that the right-hand sides of (63) and (64) are equal and hence the combinatorial lemma is proved.

By the transformation (5) we obtain for the power function $F(t) = t^k$ that $4\pi^2 \tilde{F}(1) = -\sum_{\ell=1}^{k-1} \frac{1}{\ell}$ which fits the right-hand side of (63).

Now we come to the next-to-leading term in (58), and consider for $1 \leq \ell \leq k-1$

$$\begin{aligned} &\int_{\mathbb{R}^{(k-1)d}} d\mathbf{y}_1 \cdots d\mathbf{y}_{k-1} \gamma(-\mathbf{y}_1) \cdots \gamma(-\mathbf{y}_{k-1}) \gamma(\mathbf{y}_1 + \dots + \mathbf{y}_{k-1}) \\ &\quad \times \chi_{R(\Omega-\Omega)}(\mathbf{y}_1) \cdots \chi_{R(\Omega-\Omega)}(\mathbf{y}_1 + \dots + \mathbf{y}_{k-1}) \max(0, (\mathbf{y}_1 + \dots + \mathbf{y}_{\ell}) \cdot \mathbf{n}_\mathbf{x}). \end{aligned} \quad (65)$$

Then, to leading order, we perform the integration with respect to all variables except $\mathbf{v} := \mathbf{y}_1 + \dots + \mathbf{y}_{\ell}$. This leaves us with the familiar term

$$\int_{R(\Omega-\Omega)} d\mathbf{v} |\gamma(\mathbf{v})|^2 \max(0, \mathbf{v} \cdot \mathbf{n}_\mathbf{x}) \quad (66)$$

that yields the logarithmic correction term which we know from the $k = 2$ calculation.

To complete the proof, one has to show that the error resulting from only symmetrizing the maximum function but not the product of indicator functions in (62) is of lower order as $R \rightarrow \infty$.

APPENDIX A. METHOD OF STATIONARY PHASE

We are going to cite two propositions on the method of stationary phase that will be used in this paper. To begin with, let us recall that a smooth real-valued function ϕ on \mathbb{R}^{d-1} has a critical point at $\mathbf{t}_0 \in \mathbb{R}^{d-1}$ if $\partial\phi(\mathbf{t})/\partial\mathbf{t}|_{\mathbf{t}=\mathbf{t}_0} = \mathbf{0}$. Such a point is called non-degenerate if the determinant $\det\phi_{ij}(\mathbf{t})$ of the Hessian $\phi_{ij}(\mathbf{t}) := \partial^2\phi(\mathbf{t})/\partial t_i\partial t_j$ of ϕ is non-zero at $\mathbf{t} = \mathbf{t}_0$. By $\text{sgn}\phi_{ij}(\mathbf{t})$ we denote the number of strictly positive minus the number of strictly negative eigenvalues of this Hessian at $\mathbf{t} \in \mathbb{R}^{d-1}$.

PROPOSITION 8. *Let r be a smooth complex-valued and let ϕ be a smooth real-valued function on \mathbb{R}^{d-1} . Moreover, let r have a compact support not containing a critical point of ϕ . Then*

$$\int_{\mathbb{R}^{d-1}} d\mathbf{t} r(\mathbf{t}) e^{iR\phi(\mathbf{t})} = O(R^{-N}) \quad (67)$$

as $R \rightarrow \infty$ for any $N \in \mathbb{N}$.

For a proof see [41, Chapter VIII, Section 2, Proposition 4] or [24, Theorem 7.7.1]. The second result (see [41, Chapter VIII, Section 2, Proposition 6] or [24, Theorem 7.7.5]) deals with the asymptotics of the integral in case the phase ϕ has a non-degenerate critical point.

PROPOSITION 9. *Let r be a smooth complex-valued and let ϕ be smooth real-valued function on \mathbb{R}^{d-1} . Suppose that ϕ has a non-degenerate critical point at \mathbf{t}_0 . If r is supported in a sufficiently small neighborhood of \mathbf{t}_0 (so that there is no other critical points in its support), then there exists a sequence $(z_j)_{j \in \mathbb{N}_0}$ of complex numbers such that*

$$e^{-iR\phi(\mathbf{t}_0)} \int_{\mathbb{R}^{d-1}} d\mathbf{t} r(\mathbf{t}) e^{iR\phi(\mathbf{t})} = R^{-(d-1)/2} \left(\sum_{j=0}^{N-1} z_j R^{-j} + O(R^{-N}) \right) \quad (68)$$

as $R \rightarrow \infty$ for any $N \in \mathbb{N}$. In particular, $z_0 = r(\mathbf{t}_0)(2\pi)^{(d-1)/2} |\det\phi_{ij}(\mathbf{t}_0)|^{-1/2} e^{i\frac{\pi}{4} \text{sgn}\phi_{ij}(\mathbf{t}_0)}$.

In the following $K(\mathbf{p})$ denotes the Gauss-Kronecker curvature of $\partial\Gamma$ at $\mathbf{p} \in \partial\Gamma$, and $\text{sign}(\mathbf{p})$ is the number of strictly positive minus the number of strictly negative eigenvalues of the second fundamental form of $\partial\Gamma$ at $\mathbf{p} \in \partial\Gamma$.

LEMMA 10. *Let $\Gamma \subset \mathbb{R}^d$ be a smooth, compact, d -dimensional manifold-with-boundary. Then there exists a subset E of the $(d-1)$ -dimensional unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ of full Haar measure such that for each $\mathbf{e} \in E$ there exists a non-empty and finite set $\mathcal{K}_{\mathbf{e}} \subset \partial\Gamma$ such that for all $\mathbf{k} \in \mathcal{K}_{\mathbf{e}}$*

- (i) $\mathbf{e} \cdot \mathbf{n}_{\mathbf{k}} \in \{-1, 1\}$,
- (ii) $K(\mathbf{k}) \neq 0$.

In other words, $\mathcal{K}_{\mathbf{e}}$ is the set of non-degenerate critical points of the mapping $\partial\Gamma \rightarrow \mathbb{R}, \mathbf{p} \mapsto \mathbf{p} \cdot \mathbf{e}$.

As we noted in Remark 4 for the cube $\Gamma = [-1, 1]^d$, smoothness of $\partial\Gamma$ cannot be relaxed to piecewise smoothness without jeopardizing the non-emptiness and/or finiteness of the sets $\mathcal{K}_{\mathbf{e}}$.

Proof. First we recall the definition of Gauss's spherical mapping $G : \partial\Gamma \rightarrow \mathbb{S}^{d-1}, \mathbf{p} \mapsto G(\mathbf{p}) := \mathbf{n}_{\mathbf{p}}$, and that the curvature $K(\mathbf{p})$ is given by the Jacobian determinant of G evaluated at $\mathbf{p} \in \partial\Gamma$. Then we define E as $E := G(\{\mathbf{p} \in \partial\Gamma : K(\mathbf{p}) \neq 0\})$. By compactness we have $G(\partial\Gamma) = \mathbb{S}^{d-1}$ and that the pre-image $\mathcal{K}_{\mathbf{e}} := G^{-1}(\{-\mathbf{e}, \mathbf{e}\})$ is a non-empty and finite set for each $\mathbf{e} \in E$. By Sard's Theorem the complement $\mathbb{S}^{d-1} \setminus E$ is of Haar measure 0. \square

Choice of co-ordinates: From now on we assume that for fixed $\mathbf{e} \in E$ the hypersurface $\partial\Gamma$ in \mathbb{R}^d is locally given by graphs of certain smooth functions $f^{(\mathbf{e}, m)} : U_{\mathbf{e}, m} \rightarrow \mathbb{R}$ defined on some open sets $U_{\mathbf{e}, m} \subseteq \mathbb{R}^{d-1}$ indexed by some $m \in \{1, \dots, M\}$ for some finite $M \in \mathbb{N}$ (M can be chosen independent of \mathbf{e}). To be more specific, let us assume for a moment that $\mathbf{e} = (0, \dots, 0, 1)$. Then,

up to a permutation of the d co-ordinates $\mathbf{t} = (t_1, \dots, t_{d-1})$ and $f^{(e,m)}(\mathbf{t})$, we have $\text{graph } f^{(e,m)} := \{(\mathbf{t}, f^{(e,m)}(\mathbf{t})) \in \mathbb{R}^{d-1} \times \mathbb{R} : \mathbf{t} \in U_{e,m}\}$ and $\bigcup_{m=1}^M \text{graph } f^{(e,m)} = \partial\Gamma$. In addition, we may assume that each $\text{graph } f^{(e,m)} \subseteq \partial\Gamma$ is small enough so that it contains at most one $\mathbf{k} \in \mathcal{K}_e$ (see Lemma 10) and each $\mathbf{k} \in \mathcal{K}_e$ is contained in only one of these graphs. More precisely, for every $\mathbf{k} \in \mathcal{K}_e$ there exists a unique $m_{\mathbf{k}} \in \{1, \dots, M\}$ such that $\mathbf{k} \in \text{graph } f^{(e,m_{\mathbf{k}})}$ and $\mathbf{k} = (\mathbf{t}_{\mathbf{k}}, f^{(e,m_{\mathbf{k}})}(\mathbf{t}_{\mathbf{k}}))$ for some $\mathbf{t}_{\mathbf{k}} \in U_{e,m_{\mathbf{k}}}$. Note that such a point $\mathbf{t}_{\mathbf{k}}$ is a critical point of $f^{(e,m_{\mathbf{k}})}$, that is, $\left. \frac{\partial f^{(e,m_{\mathbf{k}})}(\mathbf{t})}{\partial t} \right|_{\mathbf{t}=\mathbf{t}_{\mathbf{k}}} = \mathbf{0}$. Furthermore, the curvature of $\partial\Gamma$ at $\mathbf{k} \in \mathcal{K}_e$ becomes the determinant of the Hessian of $f^{(e,m_{\mathbf{k}})}$ at $\mathbf{t}_{\mathbf{k}}$ (that is, $K(\mathbf{k}) = \det(f_{ij}^{(e,m_{\mathbf{k}})}(\mathbf{t}_{\mathbf{k}}))$), and that $\text{sgn}(\mathbf{k}) = \text{sgn}(f_{ij}^{(e,m_{\mathbf{k}})}(\mathbf{t}_{\mathbf{k}}))$. If $e = \mathcal{R}(0, \dots, 0, 1)$ for a suitable rotation \mathcal{R} then we simply rotate the graphs of $f^{(e,m)}$ by this \mathcal{R} .

LEMMA 11 (Decay of the function $\gamma_{\mathbf{x}}$). *Let $\Gamma \subset \mathbb{R}^d$, $\mathcal{K}_e \subset \partial\Gamma$, and $E \subseteq \mathbb{S}^{d-1}$ be as in Lemma 10. Finally, let $\gamma_{\mathbf{x}}(\mathbf{v})$ be defined by (12). Then one has for large $v > 0$ the asymptotic formula*

$$\gamma_{\mathbf{x}}(\mathbf{v}) = -i(2\pi v)^{-\frac{d+1}{2}} \sum_{\mathbf{k} \in \mathcal{K}_{\mathbf{v}/v}} \frac{\text{sgn}(\mathbf{v} \cdot \mathbf{n}_{\mathbf{k}})}{\sqrt{|K(\mathbf{k})|}} \alpha(\mathbf{x}, \mathbf{k}) e^{i\mathbf{v} \cdot \mathbf{k} + \frac{i\pi}{4} \text{sign}(\mathbf{k})} \left(1 + O(v^{-1})\right) \quad (69)$$

for all $\mathbf{v}/v \in E$. The remainder term $O(v^{-1})$ is independent of $\mathbf{x} \in \Omega$.

Formula (69) is a slight variant of a standard result that can be found, for example, in [24, Theorem 7.7.14]. For this identification we note that $\text{sgn}(\mathbf{v} \cdot \mathbf{n}_{\mathbf{k}}) e^{\frac{i\pi}{4} \text{sign}(\mathbf{k})} = e^{\frac{i\pi}{4} \sigma(\mathbf{k})}$, where “Hörmander’s index” $\sigma(\mathbf{k})$ denotes the number of centers of curvature at \mathbf{k} in the direction \mathbf{v}/v minus the number of centers of curvature at \mathbf{k} in the direction $-\mathbf{v}/v$. Nevertheless, since we are using these co-ordinates in the proof of Theorem 2, we provide a proof based on Propositions 8 and 9.

Proof. As in (16) we use integration by parts to obtain

$$\begin{aligned} (2\pi)^d \gamma_{\mathbf{x}}(\mathbf{v}) &= \frac{\mathbf{v}}{iv^2} \cdot \int_{\Gamma} d\mathbf{p} \alpha(\mathbf{x}, \mathbf{p}) \frac{\partial}{\partial \mathbf{p}} e^{i\mathbf{v} \cdot \mathbf{p}} \\ &= \frac{\mathbf{v}}{iv^2} \cdot \left(\int_{\partial\Gamma} d\sigma(\mathbf{p}) \mathbf{n}_{\mathbf{p}} \alpha(\mathbf{x}, \mathbf{p}) e^{i\mathbf{v} \cdot \mathbf{p}} - \int_{\Gamma} d\mathbf{p} \left(\frac{\partial}{\partial \mathbf{p}} \alpha(\mathbf{x}, \mathbf{p}) \right) e^{i\mathbf{v} \cdot \mathbf{p}} \right). \end{aligned} \quad (70)$$

For the second integral in the last equation we may repeat the same integration-by-parts procedure with $\frac{\mathbf{v}}{v^2} \cdot \partial \alpha(\mathbf{x}, \mathbf{p}) / \partial \mathbf{p}$ instead of $\alpha(\mathbf{x}, \mathbf{p})$. This results in a term of the order $v^{-\frac{d-3}{2}}$ but with the same phase as the leading term. Therefore the leading term of $\gamma_{\mathbf{x}}(\mathbf{v})$ as $v \rightarrow \infty$ stems from the first integral, which we consider in what follows.

Now, let $e := \mathbf{v}/v$ in E be fixed. Moreover, let $(\psi_{\lambda})_{\lambda}$ be a finite C^{∞} -partition of unity which is subordinate to the covering $(\text{graph } f^{(e,m)})_m$ of $\partial\Gamma$ in the sense that for each λ , $\text{supp}(\psi_{\lambda}) \subset \text{graph } f^{(e,m(\lambda))}$ for some uniquely determined $m(\lambda) \in \{1, \dots, M\}$. In addition, if $\mathcal{K}_e \cap \text{supp}(\psi_{\lambda}) =: \{\mathbf{k}(\lambda)\}$ is non-empty for some λ , then we require that $\psi_{\lambda}(\mathbf{p}) = 1$ for all \mathbf{p} in a neighborhood of this $\mathbf{k}(\lambda)$.

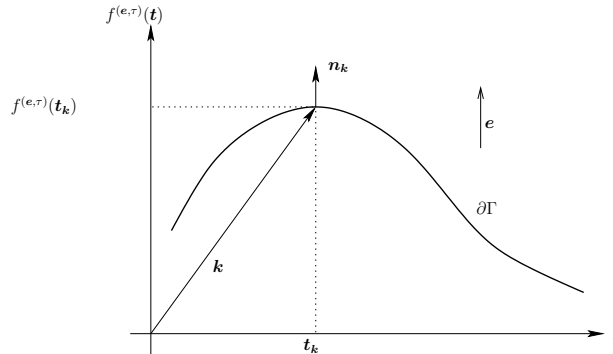


FIGURE 2. Co-ordinates for the t -integration.

In our co-ordinates we therefore get

$$\mathbf{e} \cdot \int_{\partial\Gamma} d\sigma(\mathbf{p}) \mathbf{n}_{\mathbf{p}} \alpha(\mathbf{x}, \mathbf{p}) e^{i\mathbf{v} \cdot \mathbf{p}} = \sum_{\lambda} \int_{U_{\mathbf{e}, m(\lambda)}} dt r_{\lambda}(\mathbf{t}) e^{i\mathbf{v} \phi_{\lambda}(\mathbf{t})}, \quad (71)$$

with certain smooth functions r_{λ} and ϕ_{λ} . In case $\mathbf{e} = (0, \dots, 0, 1)$, $\text{graph} f^{(\mathbf{e}, m)} := \{(\mathbf{t}, f^{(\mathbf{e}, m)}(\mathbf{t})) \in \mathbb{R}^{d-1} \times \mathbb{R} : \mathbf{t} \in U_{\mathbf{e}, m}\}$ as above (see also Figure 2), and abbreviating $\mathbf{p}^{(\lambda)}(\mathbf{t}) := (\mathbf{t}, f^{(\mathbf{e}, m(\lambda))}(\mathbf{t}))$ with $\mathbf{t} \in U_{\mathbf{e}, m(\lambda)}$, we explicitly have

$$r_{\lambda}(\mathbf{t}) = \text{sgn}(\mathbf{e} \cdot \mathbf{n}_{\mathbf{p}^{(\lambda)}(\mathbf{t})}) \mathbf{e} \cdot (-\partial f^{(\mathbf{e}, m(\lambda))}(\mathbf{t})/\partial \mathbf{t}, 1) \psi_{\lambda}(\mathbf{p}^{(\lambda)}(\mathbf{t})) \alpha(\mathbf{x}, \mathbf{p}^{(\lambda)}(\mathbf{t})), \quad (72)$$

$$\phi_{\lambda}(\mathbf{t}) = \mathbf{e} \cdot \mathbf{p}^{(\lambda)}(\mathbf{t}). \quad (73)$$

Formula (71) follows from the three facts

- (i) $\sum_{\lambda} \psi_{\lambda}(\mathbf{p}) = 1$,
- (ii) $d\sigma(\mathbf{p}) = dt \sqrt{1 + |\partial f^{(\mathbf{e}, m(\lambda))}(\mathbf{t})/\partial \mathbf{t}|^2}$ for the area measure on $\text{graph} f^{(\mathbf{e}, m(\lambda))}$, and
- (iii) $\mathbf{n}_{\mathbf{p}} = \text{sgn}(\mathbf{e} \cdot \mathbf{n}_{\mathbf{p}^{(\lambda)}(\mathbf{t})}) (-\partial f^{(\mathbf{e}, m(\lambda))}(\mathbf{t})/\partial \mathbf{t}, 1) / \sqrt{1 + |\partial f^{(\mathbf{e}, m(\lambda))}(\mathbf{t})/\partial \mathbf{t}|^2}$ for the unit normal vector at $\mathbf{p} \in \text{graph} f^{(\mathbf{e}, m(\lambda))}$.

We note that the signum function in r_{λ} takes either the value 1 or -1 on the whole of $U_{\mathbf{e}, m(\lambda)}$.

Next, we want to replace the domain of integration $U_{\mathbf{e}, m(\lambda)}$ on the right-hand side of (71) by \mathbb{R}^{d-1} without changing the value of the integral. Since ψ_{λ} has compact support in $\text{graph} f^{(\mathbf{e}, m)}$, we smoothly extend r_{λ} to \mathbb{R}^{d-1} simply by setting $r_{\lambda}(\mathbf{t}) := 0$ if $\mathbf{t} \notin U_{\mathbf{e}, m(\lambda)}$. The phase function ϕ_{λ} is smoothly extended by Urysohn's Lemma.

Now, we split the sum in (71) into a sum over those λ such that $\mathcal{K}_{\mathbf{e}} \cap \text{supp}(\psi_{\lambda}) = \emptyset$ and those for which this intersection is non-empty; in fact, it contains then only a single point, $\mathbf{k}(\lambda)$. Thus we have

$$\begin{aligned} & \frac{\mathbf{v}}{v} \cdot \int_{\partial\Gamma} d\sigma(\mathbf{k}) \mathbf{n}_{\mathbf{k}} \alpha(\mathbf{x}, \mathbf{k}) e^{i\mathbf{v} \cdot \mathbf{k}} \\ &= \sum_{\lambda: \mathcal{K}_{\mathbf{e}} \cap \text{supp}(\psi_{\lambda}) = \emptyset} \int_{\mathbb{R}^{d-1}} dt r_{\lambda}(\mathbf{t}) e^{i\mathbf{v} \phi_{\lambda}(\mathbf{t})} + \sum_{\lambda: \mathcal{K}_{\mathbf{e}} \cap \text{supp}(\psi_{\lambda}) = \{\mathbf{k}(\lambda)\}} \int_{\mathbb{R}^{d-1}} dt r_{\lambda}(\mathbf{t}) e^{i\mathbf{v} \phi_{\lambda}(\mathbf{t})}. \end{aligned} \quad (74)$$

In the first sum we get the decay v^{-N} for any N according to Proposition 8. In the second sum, ϕ_{λ} is expanded to second order around its critical point $\mathbf{k}(\lambda)$. By Proposition 9 we therefore arrive at (69). By compactness of Ω we may choose the remainder term $O(v^{-1})$ to be independent of $\mathbf{x} \in \Omega$. \square

APPENDIX B. ROCCAFORETE'S ESTIMATE ON THE VOLUME OF SELF-INTERSECTIONS

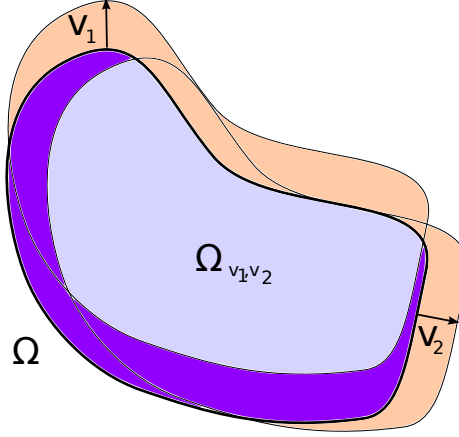
In [35, Theorem 2.1], Roccaforte proved a theorem which is (by one order of ε below) more precise than what we need here. See also a previous version by Widom [47, Lemma 2 & 2']). But Roccaforte's proof also allows for the inclusion of a smooth function in the integrand. For the convenience of the reader we present his proof almost literally and do not claim any originality. Note, however, that the derivative of f effects the correction of the order ε^2 but this is not needed here.

THEOREM 12 (Roccaforte). *Let $\Omega \subset \mathbb{R}^d$ be a compact set with C^2 -boundary $\partial\Omega$, $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^d$, $\varepsilon > 0$, and $\Omega_{\mathbf{v}_1, \dots, \mathbf{v}_n} := \Omega \cap (\Omega + \mathbf{v}_1) \cap \dots \cap (\Omega + \mathbf{v}_n)$. Let f be a C^1 -function defined on Ω . Then there exists a constant C depending on Ω and (the supremum of the derivative of) f so that*

$$\left| \int_{\Omega \setminus \Omega_{\varepsilon \mathbf{v}_1, \dots, \varepsilon \mathbf{v}_n}} d\mathbf{x} f(\mathbf{x}) + \varepsilon \int_{\partial\Omega} d\sigma(\mathbf{x}) f(\mathbf{x}) \max_{1 \leq k \leq n} (0, \mathbf{v}_k \cdot \mathbf{n}_{\mathbf{x}}) \right| \leq C \varepsilon^2, \quad (75)$$

where $\mathbf{n}_{\mathbf{x}}$ is the unit outward normal at $\mathbf{x} \in \partial\Omega$.

Proof. Let $\Omega_{\varepsilon} := \Omega_{\varepsilon \mathbf{v}_1, \dots, \varepsilon \mathbf{v}_n}$. Since $\partial\Omega$ is compact there exists an ε_0 and an ε_0 -tubular neighborhood N_{ε_0} of $\partial\Omega$ such that each $\mathbf{x} \in N_{\varepsilon_0}$ can be written uniquely as $\mathbf{x} = \bar{\mathbf{x}} - s \mathbf{n}_{\bar{\mathbf{x}}}$, where $\bar{\mathbf{x}} \in \partial\Omega$, $\mathbf{n}_{\bar{\mathbf{x}}}$ is the unit outward normal vector at $\bar{\mathbf{x}}$, and $|s| < \varepsilon_0$. If ε is small enough then $\Omega \setminus \Omega_{\varepsilon} \subset N_{\varepsilon_0}$.


 FIGURE 3. The integration region $\Omega \setminus \Omega_\varepsilon$ is marked in dark purple.

Let $\{U_j, \psi_j\}$ be a finite atlas of co-ordinate neighborhoods covering $\partial\Omega$. Let $V_j := \{\mathbf{x} \in N_{\varepsilon_0} : \text{if } \mathbf{x} = \bar{\mathbf{x}} - s\mathbf{n}_{\bar{\mathbf{x}}}, \text{ then } \bar{\mathbf{x}} \in U_j\}$. Define $\phi_j : V_j \rightarrow \mathbb{R}^{d-1} \times \mathbb{R}$ as follows: if $\mathbf{x} = \bar{\mathbf{x}} - s\mathbf{n}_{\bar{\mathbf{x}}} \in V_j$ and $\psi_j(\bar{\mathbf{x}}) = \bar{\mathbf{u}} \in \mathbb{R}^{d-1}$, then $\phi_j(\mathbf{x}) := \bar{\mathbf{u}} + s\mathbf{n}$, where $\mathbf{n} := (0, \dots, 0, 1)$ is the unit vector in $\mathbb{R}^{d-1} \times \mathbb{R}$ normal to \mathbb{R}^{d-1} . By the compactness of $\partial\Omega$ there exist open sets $N_j \subset V_j$ such that the N_j are an open cover of N_{ε_0} and the distance from N_j to the complement of V_j is, for all j , greater than some ε_1 . If ε is chosen such that $\max\{|\varepsilon \mathbf{v}_k|, 1 \leq k \leq n\} < \varepsilon_1$ then, for all $j, k, \mathbf{x} \in N_j$ implies $\mathbf{x} - \varepsilon \mathbf{v}_k \in V_j$. Let $W_j := N_j \cap \partial\Omega \subset U_j$.

Let $\{\rho_j\}$ be a partition of unity subordinate to the cover $\bigcup_j W_j$ of $\partial\Omega$. Each ρ_j extends to a function $\tilde{\rho}_j$ on $N_j \cap \Omega \setminus \Omega_\varepsilon$ by defining $\tilde{\rho}_j(\bar{\mathbf{x}} - s\mathbf{n}_{\bar{\mathbf{x}}}) := \rho_j(\bar{\mathbf{x}})$. It now suffices to prove

$$\int_{N_j \cap \Omega \setminus \Omega_\varepsilon} d\mathbf{x} f(\mathbf{x}) \tilde{\rho}_j(\mathbf{x}) + \varepsilon \int_{W_j} d\sigma(\bar{\mathbf{x}}) f(\bar{\mathbf{x}}) \rho_j(\bar{\mathbf{x}}) \max_{1 \leq k \leq n} (0, \mathbf{v}_k \cdot \mathbf{n}_{\bar{\mathbf{x}}}) = O(\varepsilon^2). \quad (76)$$

In what follows the index j will be dropped. From the construction of ϕ it follows that for $\mathbf{y} \in V$, $\mathbf{y} \in \Omega \cap V$ if and only if $\phi(\mathbf{y}) \cdot \mathbf{n} = s \geq 0$. Hence for $\mathbf{x} \in N$, $\mathbf{x} \in N \cap \Omega_\varepsilon$ if and only if $s \geq 0$ and, for all $1 \leq k \leq n$, $\phi(\mathbf{x} - \varepsilon \mathbf{v}_k) \cdot \mathbf{n} \geq 0$. By Taylor's Theorem $\mathbf{x} \in N \cap \Omega_\varepsilon$ if and only if $s \geq 0$ and, for all k ,

$$\phi(\mathbf{x}) \cdot \mathbf{n} - \varepsilon (D_{\mathbf{x}}\phi)(\mathbf{v}_k) \cdot \mathbf{n} + R(\varepsilon) \geq 0, \quad (77)$$

where $R(\varepsilon) = O(\varepsilon^2)$, the estimate being uniform over \mathbf{x} since $\phi \in C^2(V)$; $D_{\mathbf{x}}\phi$ denotes the derivative of ϕ at \mathbf{x} with matrix elements $(D_{\mathbf{x}}\phi)_{ij} = \frac{\partial \phi_i}{\partial x_j}(\mathbf{x})$.

Next we show that for any $\mathbf{v} \in \mathbb{R}^d$ and all $\mathbf{x} = \bar{\mathbf{x}} - s\mathbf{n}_{\bar{\mathbf{x}}} \in N$,

$$(D_{\mathbf{x}}\phi)(\mathbf{v}) \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}_{\bar{\mathbf{x}}}. \quad (78)$$

To see this, let $\mathbf{v}_t := \mathbf{v} - (\mathbf{v} \cdot \mathbf{n}_{\bar{\mathbf{x}}})\mathbf{n}_{\bar{\mathbf{x}}}$ be the component of \mathbf{v} that is tangent to $\partial\Omega$ at \mathbf{x} . Writing

$$(D_{\mathbf{x}}\phi)(\mathbf{v}) \cdot \mathbf{n} = (D_{\mathbf{x}}\phi)(\mathbf{v}_t) \cdot \mathbf{n} + (\mathbf{v} \cdot \mathbf{n}_{\bar{\mathbf{x}}}) (D_{\mathbf{x}}\phi)(\mathbf{n}_{\bar{\mathbf{x}}}) \cdot \mathbf{n} \quad (79)$$

it suffices to prove

$$(D_{\mathbf{x}}\phi)(\mathbf{v}_t) \cdot \mathbf{n} = 0 \quad \text{and} \quad (D_{\mathbf{x}}\phi)(\mathbf{n}_{\bar{\mathbf{x}}}) = \mathbf{n}. \quad (80)$$

For each fixed s_0 with $|s_0| < \varepsilon_0$ the map $\phi^{-1} : (\bar{\mathbf{u}}, s_0) \mapsto \bar{\mathbf{x}}(\bar{\mathbf{u}}) - s_0 \mathbf{n}_{\bar{\mathbf{x}}}(\bar{\mathbf{u}})$ describes the hypersurface $W - s_0 \mathbf{n}_{\bar{\mathbf{x}}}$. The vectors $\partial(\bar{\mathbf{x}} - s_0 \mathbf{n}_{\bar{\mathbf{x}}})/\partial u_i(\bar{\mathbf{u}}_0)$ form a basis for the tangent space to $W - s_0 \mathbf{n}_{\bar{\mathbf{x}}}$ at $\bar{\mathbf{x}}(\bar{\mathbf{u}}_0) - s_0 \mathbf{n}_{\bar{\mathbf{x}}}(\bar{\mathbf{u}}_0)$. Thus, the derivative $D_{(\bar{\mathbf{u}}_0, s_0)}\phi^{-1}$ sends vectors $(\bar{\mathbf{u}}, 0)$ to vectors tangent to $W - s_0 \mathbf{n}_{\bar{\mathbf{x}}}$ at $\bar{\mathbf{x}}(\bar{\mathbf{u}}_0) - s_0 \mathbf{n}_{\bar{\mathbf{x}}}(\bar{\mathbf{u}}_0)$ and sends \mathbf{n} to $\mathbf{n}_{\bar{\mathbf{x}}}$. Hence it suffices to show that for any $\mathbf{x} = \bar{\mathbf{x}} - s\mathbf{n}_{\bar{\mathbf{x}}}$, the tangent space $T_{\mathbf{x}}(W - s\mathbf{n}_{\bar{\mathbf{x}}}) = T_{\bar{\mathbf{x}}}(\partial\Omega)$. But

$$\mathbf{n}_{\bar{\mathbf{x}}} \cdot \frac{\partial}{\partial u_i}(\bar{\mathbf{x}} - s\mathbf{n}_{\bar{\mathbf{x}}}) = -s\mathbf{n}_{\bar{\mathbf{x}}} \cdot \frac{\partial}{\partial u_i}(\mathbf{n}_{\bar{\mathbf{x}}}) = -\frac{s}{2} \frac{\partial}{\partial u_i}(\mathbf{n}_{\bar{\mathbf{x}}} \cdot \mathbf{n}_{\bar{\mathbf{x}}}) = 0.$$

So, $T_{\mathbf{x}}(W - s\mathbf{n}_{\bar{\mathbf{x}}})$ is orthogonal to $\mathbf{n}_{\bar{\mathbf{x}}}$ and hence the same as the tangent space $T_{\bar{\mathbf{x}}}(\partial\Omega)$.

From (77) and (78), $\mathbf{x} \in N \cap \Omega \setminus \Omega_\varepsilon$ if and only if

$$s \geq 0 \quad \text{and, for some } 1 \leq k \leq n, \quad s - \varepsilon \mathbf{v}_k \cdot \mathbf{n}_{\bar{\mathbf{x}}} + R(\varepsilon) < 0. \quad (81)$$

Let

$$S_\varepsilon := \{\mathbf{x} \in N : s \geq 0 \text{ and for all } 1 \leq k \leq n, s - \varepsilon \mathbf{v}_k \cdot \mathbf{n}_{\bar{\mathbf{x}}} + R(\varepsilon) \geq 0\},$$

and for real δ ,

$$I_\delta := \{\mathbf{x} \in N : s \geq 0 \text{ and for all } 1 \leq k \leq n, s - \varepsilon \mathbf{v}_k \cdot \mathbf{n}_{\bar{\mathbf{x}}} \geq \delta \varepsilon^2\}.$$

Then we have

$$N \cap \Omega \setminus \Omega_\varepsilon = N \cap \Omega \setminus S_\varepsilon, \quad (82)$$

and the volume of the symmetric difference,

$$|S_\varepsilon \triangle I_0| = O(\varepsilon^2). \quad (83)$$

(82) is obvious. To prove (83) note first that since $R(\varepsilon) = O(\varepsilon^2)$ there is for each $\delta > 0$ an $\varepsilon_\delta > 0$ so that

$$\varepsilon < \varepsilon_\delta \quad \text{implies} \quad I_\delta \subset S_\varepsilon \subset I_{-\delta}. \quad (84)$$

Next,

$$\varepsilon < \varepsilon_\delta \quad \text{implies} \quad S_\varepsilon \triangle I_0 \subset I_{-\delta} \setminus I_\delta. \quad (85)$$

For, if $\mathbf{x} \in S_\varepsilon$ but $\mathbf{x} \notin I_0$ we have by (84) that $\mathbf{x} \in I_{-\delta}$ and $\mathbf{x} \notin I_0$ implies $\mathbf{x} \notin I_\delta$. Similarly, if $\mathbf{x} \notin S_\varepsilon$ but $\mathbf{x} \in I_0$ implies $\mathbf{x} \in I_{-\delta} \setminus I_\delta$. From (85) it then follows that for $\varepsilon < \varepsilon_\delta$,

$$|S_\varepsilon \triangle I_0| \leq |I_{-\delta} \setminus I_\delta| = \int_{\phi(I_{-\delta} \setminus I_\delta)} d\bar{\mathbf{u}} ds \left| \det D_{(\bar{\mathbf{u}}, s)} \phi^{-1} \right|.$$

But

$$\phi(I_{-\delta} \setminus I_\delta) = \{\phi(\mathbf{x}), \mathbf{x} \in N : s \geq 0, -\delta \varepsilon^2 + \varepsilon \max_{1 \leq k \leq n} (\mathbf{v}_k \cdot \mathbf{n}_{\bar{\mathbf{x}}}) \leq s \leq \delta \varepsilon^2 + \varepsilon \max_{1 \leq k \leq n} (\mathbf{v}_k \cdot \mathbf{n}_{\bar{\mathbf{x}}})\}. \quad (86)$$

Thus the above integral is

$$\leq 2\delta \varepsilon^2 \sup_{(\bar{\mathbf{u}}, s) \in \phi(N)} \left| \det D_{(\bar{\mathbf{u}}, s)} \phi^{-1} \right| \int_{\phi(\partial\Omega \cap I_{-\delta} \setminus I_\delta)} d\bar{\mathbf{u}} \leq 2\delta \varepsilon^2 M,$$

where $M := \sup\{|\det D_{(\bar{\mathbf{u}}, s)} \phi^{-1}| : (\bar{\mathbf{u}}, s) \in \phi(N)\} |\phi(W)|$ and $|\phi(W)|$ is the $(d-1)$ -dimensional Lebesgue volume of $W \subset \mathbb{R}^{d-1}$; by the compactness of Ω one can guarantee $M < \infty$. This shows (83).

This allows us to replace $N \cap \Omega \setminus \Omega_\varepsilon$ by $N \cap \Omega \setminus I_0$ in (76). Changing variables we obtain,

$$\int_{\Omega \cap N \setminus I_0} d\mathbf{x} (f\tilde{\rho})(\mathbf{x}) = \int_{\phi(\Omega \cap N \setminus I_0)} d\bar{\mathbf{u}} ds (f\tilde{\rho}) \circ \phi^{-1}(\bar{\mathbf{u}}, s) \left| \det D_{(\bar{\mathbf{u}}, s)} \phi^{-1} \right|. \quad (87)$$

Now we expand $(f\tilde{\rho}) \circ \phi^{-1}$ and $|\det D\phi^{-1}|$ at $s = 0$ to first order. Then the last integral equals

$$\int_{\phi(\Omega \cap N \setminus I_0)} d\bar{\mathbf{u}} ds [(f\tilde{\rho}) \circ \phi^{-1}(\bar{\mathbf{u}}, 0) |\det D_{(\bar{\mathbf{u}}, 0)} \phi^{-1}| + O(s)], \quad (88)$$

where by the definitions of ϕ and I_0

$$\phi(\Omega \cap N \setminus I_0) = \{\phi(\mathbf{x}), \mathbf{x} \in N : 0 \leq s \leq \varepsilon \max_{1 \leq k \leq n} (0, \mathbf{v}_k \cdot \mathbf{n}_{\bar{\mathbf{x}}})\}. \quad (89)$$

Integrating with respect to s yields

$$\varepsilon \int_{\phi(W)} d\bar{\mathbf{u}} (f\rho) \circ \phi^{-1}(\bar{\mathbf{u}}, 0) |\det D_{(\bar{\mathbf{u}}, 0)} \phi^{-1}| \max_{1 \leq k \leq n} (0, \mathbf{v}_k \cdot \mathbf{n}_{\bar{\mathbf{x}}}) + O(\varepsilon^2), \quad (90)$$

which proves our statement by another change of variables. \square

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ARNOLD SOMMERFELD CENTER, LUDWIG–MAXIMILIANS–UNIVERSITÄT MÜNCHEN, THERESIENSTRASSE 37, 80333 MÜNCHEN

E-mail address: `helling@atdote.de`

INSTITUT FÜR THEORETISCHE PHYSIK, UNIVERSITÄT ERLANGEN–NÜRNBERG, STAUDTSTRASSE 7, 91058 ERLANGEN

E-mail address: `hajo.leschke@physik.uni-erlangen.de`

E-mail address: `wolfgang.spitzer@physik.uni-erlangen.de`